

**GENERIC VANISHING AND THE GEOMETRY OF
IRREGULAR VARIETIES IN POSITIVE
CHARACTERISTIC**

by

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ABSTRACT

Vanishing theorems play a paramount role in modern birational geometry. Over fields of characteristic zero, the classical vanishing theorems of Kodaira, Nakano and Kawamata-Viehweg show that under positivity assumptions on a line bundle L , one can guarantee that the higher cohomology groups $H^i(X, \omega_X \otimes L)$ of a smooth projective variety are zero. If the positivity assumptions are dropped (namely, if the line bundles involved have zero first Chern class), a celebrated theorem of Green and Lazarsfeld shows that one can still obtain vanishing of certain cohomology groups for an open subset of topologically trivial line bundles.

Generic vanishing theorems have proved to be remarkably useful, and among their many applications, one can highlight Ein and Lazarsfeld's results on the birational geometry of irregular varieties or the study of pluricanonical maps of varieties of maximal Albanese dimension.

Over fields of positive characteristic, a theorem of Hacon and Kovács shows that the obvious extension of Green and Lazarsfeld's generic vanishing theorem is false. Notwithstanding this, recent work of Hacon and Patakfalvi provides a generic vanishing statement in positive characteristic which, albeit necessarily weaker, is strong enough to prove positive characteristic versions of Kawamata's celebrated characterization of abelian varieties.

The objective of this dissertation is twofold. In the first place, we generalize Hacon and Patakfalvi's generic vanishing theorem in the context of Pareschi and Popa's generic vanishing theory. Once the generic vanishing groundwork is laid down, we apply it to derive relevant geometric consequences. Concretely, we prove some results of Ein and Lazarsfeld on the geometry of irregular varieties and on the singularities of Theta divisors and we also prove some statements that might shed some light on the problem of the rationality of the tricanonical map of varieties of general type and of maximal Albanese dimension in positive characteristic.

To Lin, Carme and Carol.

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CHAPTER 1

INTRODUCTION

Let X be a smooth projective variety over an algebraically closed field k . If X admits a generically finite morphism to an abelian variety $f : X \rightarrow A$ we say that X has maximal Albanese dimension (m.A.d.). Over the complex numbers, the geometry of projective varieties of maximal Albanese dimension is well understood and its study relies primarily on the generic vanishing theorem of Green and Lazarsfeld, further developed by Chen, Hacon, Popa, Pareschi, Schnell, Simpson and others.

Until fairly recently, little was known about generic vanishing and the geometry of varieties of maximal Albanese dimension over fields of positive characteristic. In fact, Hacon and Kovács showed in [14] that the obvious positive characteristic extension of the generic vanishing theorem does not hold. In a recent paper [15], Hacon and Patakfalvi shed further light into this problem by proving a generic vanishing statement in positive characteristic which, albeit necessarily weaker than its characteristic zero counterpart, was still strong enough to imply positive characteristic versions of celebrated results of Kawamata on the characterization of abelian varieties (c.f. [15], [16]).

Our goal in this dissertation is twofold. In the first place, we will be generalizing Hacon and Patakfalvi's generic vanishing theorem in the context of Pareschi and Popa's generic vanishing theory, in a sense to be made precise shortly. After laying down the generic vanishing groundwork, we shall apply it to derive geometric consequences from it. Concretely, we will be proving some results of Ein and Lazarsfeld on the geometry of irregular varieties and on the singularities of Theta divisors (c.f. [10]) and we will also prove some statements that might shed some light on the problem of the rationality of the tricanonical map of varieties of general type and of maximal Albanese dimension in positive characteristic.

In this introduction, we briefly review the generic vanishing theory over the complex numbers and we illustrate its power by presenting some of its major geometric consequences. We will be sketching the proofs of some of the latter from Pareschi and Popa's perspective (c.f. [30], [34]) since our positive characteristic extensions rely on similar ideas.

1.1 Generic vanishing over the complex numbers

In order to motivate the notion of generic vanishing, let us recall that the importance of vanishing theorems in algebraic geometry relies on the fact that many geometric questions can be formulated in terms of coherent sheaves, their global sections and their cohomology. For instance, given a short exact sequence of coherent sheaves

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$$

on an algebraic variety X , the vanishing of the cohomology group $H^1(X, \mathcal{F}')$ provides a sufficient condition for any section of \mathcal{F}'' to lift to a section of \mathcal{F} . Similarly, given a coherent sheaf \mathcal{F} on an algebraic variety X , if all cohomology groups $H^i(X, \mathcal{F})$ vanish for $i > 0$, we have that $H^0(X, \mathcal{F}) = \chi(X, \mathcal{F})$. Computing the Euler characteristic hence provides a way of showing existence of sections.

Vanishing theorems provide conditions under which certain cohomology groups vanish, most commonly positivity conditions of some line bundle. For instance, Kodaira's vanishing theorem states that if L is an ample line bundle on a smooth projective variety, then $H^i(X, \omega_X \otimes L) = 0$ for all $i > 0$ and, under the same conditions, Nakano's vanishing theorem states that $H^p(X, \Omega_X^q \otimes L) = 0$ whenever $p + q > \dim X$. The positivity assumption on the line bundle L can be relaxed: Kawamata-Viehweg's vanishing theorem, which lies at the heart of modern algebraic geometry, states that if L is a big and nef line bundle on a smooth projective variety, then the conclusion of Kodaira's vanishing theorem still holds.

In order for the above vanishing theorem to hold, we need a line bundle with positive first Chern class $c_1(L) > 0$ and the theory of generic vanishing originated in the study of the leftover case $c_1(L) = 0$. Line bundles with vanishing first class are the so-called *topologically trivial* line bundles, since their underlying smooth line bundle is trivial, and they are parameterized by the points of $\text{Pic}^0(X)$.

Unlike the case of positive line bundle, one cannot possibly expect to derive a vanishing theorem as above that works for all topologically trivial line bundles $L \in \text{Pic}^0(X)$, since there may be special line bundles whose cohomology cannot vanish for geometrical reasons. We illustrate this statement with an example due to Beauville, taken from [37]. Let X be a smooth projective variety equipped with a morphism $f : X \rightarrow C$ with connected fibers (hence $f_*\mathcal{O}_X \simeq \mathcal{O}_C$) to a curve of genus $g \geq 2$. Let $L \in \text{Pic}^0(C)$. From Leray's spectral sequence, the group $H^1(X, f^*L)$ sits in a short exact sequence

$$0 \rightarrow H^1(C, L) \rightarrow H^1(X, f^*L) \rightarrow H^0(C, R^1f_*\mathcal{O}_X \otimes L) \rightarrow 0$$

Since the curve C has genus at least 2, by Serre duality, we have $H^1(C, L) \simeq H^0(C, \omega_X \otimes L^\vee) \neq 0$ for every $L \in \text{Pic}^0(C)$. The short exact sequence then implies that $\text{Pic}^0(X)$ contains a whole subvariety - isomorphic to $\text{Pic}^0(C)$ - of line bundles whose first cohomology is nontrivial. Furthermore, even if we exclude those special line bundles, the top cohomology group $H^{\dim X}(X, L)$ will typically be non-zero.

We now formulate the original generic vanishing statement of Green and Lazarsfeld. Let X be a smooth projective variety over \mathbb{C} and let $a : X \rightarrow \text{Alb}(X)$ be the Albanese morphism, which is characterized by the universal property that $a^* : \text{Pic}^0(\text{Alb}(X)) \rightarrow \text{Pic}^0(X)$ is an isomorphism. Denoting by

$$V^i(\omega_X) = \{P \in \text{Pic}^0(X) \mid h^i(X, \omega_X \otimes P) \neq 0\}$$

the cohomology support loci, Green and Lazarsfeld theorem can be stated as follows.

Theorem 1 ([11], [12], [40]). *Every irreducible component of $V^i(\omega_X)$ is a (torsion) translate of a (reduced) subtorus of $\text{Pic}^0(X)$ of codimension at least $i - \dim(X) - \dim(a(X))$. If $\dim(X) = \dim a(X)$, then there are inclusions*

$$V^0(\omega_X) \supset V^1(\omega_X) \supset \dots \supset V^{\dim(X)}(\omega_X) = \{\mathcal{O}_X\}$$

In particular, if X is a variety of maximal Albanese dimension, then the $V^i(\omega_X)$ have codimension $\geq i$, which implies the vanishing of $H^i(X, \omega_X \otimes P)$ for $i > 0$ and generic $P \in \text{Pic}^0(X)$. This is why the above result is commonly known as the *generic vanishing theorem* and it implies that for varieties of maximal Albanese dimension, the ample line bundle in the statement of Kodaira vanishing theorem can be replaced by a general topologically trivial line bundle.

The theorem was first proven in [12] using Hodge theory. An alternative point of view using the Fourier-Mukai transforms $R\hat{S} = Rp_{\hat{A},*}(p_A^*(\cdot) \otimes \mathcal{L})$ and $RS = Rp_{A,*}(p_{\hat{A}}^*(\cdot) \otimes \mathcal{L})$ emerged in [13] and 4 [33]. Specifically, in [33], Pareschi and Popa proved the following theorem.

Theorem 2 (c.f. Theorem 3.7 in [33]). *Let A be a g -dimensional abelian variety and let \mathcal{F} be a coherent sheaf on A . The following are equivalent:*

1. *For any sufficiently ample line bundle L on \hat{A} , $H^i(A, \mathcal{F} \otimes \hat{L}^\vee) = 0$ for any $i > k$, where $\hat{L} = R^0S(L) = RS(L)$,*
2. *$R^i\hat{S}(D_A(\mathcal{F})) = 0$ for any $i < -k$,*

3. $\text{codim Supp } R^i \hat{S}(\mathcal{F}) \geq i - k$ for any $i \geq 0$, and
4. $\text{codim } V^i(\mathcal{F}) \geq i - k$ for any $i \geq 0$.

The proof relies on the following result of Hacon:

Theorem 3 (c.f. [13]). *Let \mathcal{L} be the Poincaré line bundle on $A \times \hat{A}$. If the Albanese image $a(X) \subset A$ has dimension $d - k$, then*

$$R^i p_{\hat{A},*}(\mathcal{L}_X) = 0$$

for $i \notin [d - k, k]$, where $\mathcal{L}_X = (a \times \text{id}_{\hat{A}})^* \mathcal{L}$.

In a nutshell, Pareschi and Popa show in [33] that the conclusions of Theorems 2 and 3 are equivalent. Denoting by $D_A(\cdot) = R\mathcal{H}om(\cdot, \mathcal{O}_A[\dim A])$, it follows from Grothendieck's spectral sequence, the projection formula and Grothendieck duality that

$$Rp_{\hat{A},*}(\mathcal{L}_X) \simeq R\hat{S}(D_A(Ra_*\omega_X[d])).$$

By Kollár's splitting theorem, we have $Ra_*\omega_X = \sum_{j=0}^k R^j a_*\omega_X[-j]$, so the conclusion of Theorem 3 is equivalent to the vanishing

$$\mathcal{H}^i \left(R\hat{S}(D_A(R^j a_*\omega_X)) \right) = 0, \quad 0 \leq j \leq k, \quad i < j - k$$

which is part (2) in Theorem 2.

The usefulness of the notion of GV-sheaf stems from the fact that some features of the cohomology groups $H^i(A, \mathcal{F} \otimes P_\alpha)$ and of the corresponding cohomological support loci $V^i(\mathcal{F})$ can be detected by local and sheaf-theoretic properties of the transform $RS(D_A(\mathcal{F}))$. We now proceed to illustrate some of these properties, following [30] and, once we have them recorded, we will see how to use them to derive geometric consequences.

1.2 Generic vanishing and the Fourier-Mukai transform

Let \mathcal{F} be a sheaf on a g -dimensional abelian variety A , which satisfies any of the equivalent conditions in Theorem 2. Assume for simplicity that $k = 0$ and let us refer to such a sheaf \mathcal{F} as a GV-sheaf for short. By condition 2 Theorem 2, the cohomologies of the complex $RS(D_A(\mathcal{F}))$ are all zero in degrees other than 0 and we adopt Pareschi and Popa's notation to refer to only non-zero cohomology $RS(\widehat{D_A(\mathcal{F})}) = R^0 S(D_A(\mathcal{F}))$.

Using Grothendieck duality and the cohomology and base change theorem, one can derive the following:

Lemma 4 (c.f. Proposition 1.6 in [30]). *The sheaf $RS(\widehat{D_A(\mathcal{F})})$ satisfies the following basic properties:*

1. *The rank of $RS(\widehat{D_A(\mathcal{F})})$ equals the Euler characteristic $\chi(\mathcal{F})$.*
2. *$\mathcal{E}xt_{\mathcal{O}_{\hat{A}}}^i(RS(\widehat{D_A(\mathcal{F})}), \mathcal{O}_{\hat{A}}) \simeq (-1_{\hat{A}})^* R^i \hat{S}(\mathcal{F})$.*

A crucial result in characteristic zero is the following:

Lemma 5 (c.f. Lemma 1.8 in [30]). *Let \mathcal{F} be a GV-sheaf on an abelian variety A , let W be an irreducible component of $V^0(\mathcal{F}) = \{\alpha \in \hat{A} : H^0(A, \mathcal{F} \otimes P_\alpha) \neq 0\}$ and suppose that $k = \text{codim}_{\hat{A}} W$. Then W is also a component of $V^k(\mathcal{F})$. In particular, if α is an isolated point of $V^0(\mathcal{F})$, then it is also an isolated point of $V^{\dim A}(\mathcal{F})$*

It thus follows that if \mathcal{F} is a GV-sheaf on A , then either $V^0(\mathcal{F}) = \hat{A}$ or there is a positive i such that $\text{codim}_{\hat{A}} V^i(\mathcal{F}) = 0$. By Serre duality and cohomology and base change, the condition that $V^0(\mathcal{F}) = \hat{A}$ is a proper subvariety is equivalent to the sheaf $RS(\widehat{D_A(\mathcal{F})})$ being a torsion sheaf on \hat{A} , and we have just noted that this implies the existence of a positive i for which the lower bound of Theorem 2(4) is attained. In characteristic zero, it turns out that the converse is also true, so we have:

Lemma 6 (c.f. Theorem 1.10 in [30]). *Let \mathcal{F} be a sheaf on an abelian variety A . The following are equivalent:*

1. *There is an $i > 0$ such that $\text{codim}_{\hat{A}} V^i(\mathcal{F}) = i$.*
2. *The sheaf $RS(\widehat{D_A(\mathcal{F})})$ is not torsion-free.*

1.3 The geometry of varieties of varieties of maximal Albanese dimension

We next proceed to illustrate the power of this machinery by deriving geometric consequences. We will start by using the results above to prove two celebrated results of Ein and Lazarsfeld on the geometry of irregular varieties and the singularities of Theta divisors of PPAV's (c.f. [10]). For this, we will follow [34]. We will next sketch a proof of the birationality of the tricanonical map for varieties of general type and maximal Albanese dimension which uses the same set of tools (c.f. [24]).

We start with the following theorem:

Theorem 7. (c.f. [10, Theorem 3]; [34, Theorem E]) *If X is a smooth projective variety of maximal Albanese dimension over a field of characteristic zero and $\chi(\omega_X) = 0$, then the image of the Albanese map is ruled by positive-dimensional subtori of A .*

Proof. We sketch the proof given in [34] Theorem E.

If $\chi(\omega_X) = 0$, it follows from Proposition 6 that $V^0(\omega_X) \subset \hat{A}$ is a proper subvariety. Choose an irreducible component $W \subset V^0(\omega_X)$ of codimension $p > 0$, which by Theorem 1 is a torsion translate of an abelian subvariety of \hat{A} and which we also denote by W . Let $\pi : A \rightarrow \hat{W}$ denote the dual projection and consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow{a} & Y := a(X) & \hookrightarrow & A \\ & & \downarrow h = \pi|_Y & \swarrow \pi & \\ & & \hat{W} & & \end{array}$$

Since the fibers of the projection $A \rightarrow \hat{W}$ are abelian subvarieties of dimension p , the conclusion of the theorem will follow provided that $f \geq p$, where f denotes the dimension of the generic fiber of h . In order to see this, recall the following standard facts:

1. $a_*\omega_X$ is a GV-sheaf on $Y = a(X)$ and $V^0(\omega_X) = V^0(a_*\omega_X)$.
2. By Lemma 5, if W is an irreducible component of $V^0(a_*\omega_X)$ of codimension p , then it is also a component of $V^p(a_*\omega_X)$.
3. By Theorem 5.8 in [34], $a_*\omega_X$ is a GV_{-f} -sheaf with respect to the Fourier-Mukai functor with kernel $(\pi \times 1_W)^*\mathcal{P}^{W \times \hat{W}}$, so in particular $\text{codim}_W V^p(a_*\omega_X) \geq p - f$ for every $p \geq 0$.

By (b), we have $W \subseteq V^p(a_*\omega_X) \subseteq W$ so that $\text{codim}_W V^p(a_*\omega_X) = 0$, and finally (c) yields $f \geq p$, which is what we sought to show. \square

We now move to another result of Ein and Lazarsfeld on the singularities of Theta divisors.

Theorem 8. ([10, Theorem 1] or [30, Theorem 3.5]) *Let A be an abelian variety and let $\Theta \subset A$ be a principal polarization (i.e. an ample divisor such that $h^0(A, \mathcal{O}_A(\Theta)) = 1$). If Θ is irreducible, then it is normal and has rational singularities.*

Proof. We sketch the proof given in [30] (Theorem 3.5). The conclusion of the theorem is captured by the trace map of a resolution of singularities: if $\mathfrak{a} : X \rightarrow \Theta$ is a resolution of

singularities, the fact that Θ is normal and has rational singularities is equivalent to the fact that the trace map $Tr : a_*\omega_X \rightarrow \omega_\Theta$ is an isomorphism. Bearing this in mind, Pareschi and Popa break Ein and Lazarsfeld's argument into the following steps:

1. If Θ is irreducible, then X is of general type. This relies on a classical argument due to Ueno characterizing the Itaka fibration and the Kodaira dimension of an irreducible subvariety of an abelian variety.
2. If X is of general type, then $\chi(\omega_X) > 0$. More concretely, if X is a smooth projective variety of maximal Albanese dimension and $\chi(X, \omega_X) = 0$, then the image of the Albanese map is fibred by tori (c.f. Theorem 7).
3. If the trace map was not an isomorphism, a simple computation would show that $\chi(\omega_X) = 0$, so by (i), Θ would be fibred by subtori of A , and this would contradict (i).

□

Remark 9. *Ein and Lazarsfeld's original proof invokes properties of adjoint ideals. Given any log resolution $\mu : \tilde{A} \rightarrow A$ of the pair (A, Θ) and writing $\mu^*\Theta = \Theta' + F$ with Θ' smooth and F μ -exceptional, one may define $\text{adj}(A, \Theta) = \mu_*\mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - F)$. Standard arguments show that $\text{adj}(A, \Theta) = \mathcal{O}_A$ is equivalent to Θ being normal and having rational singularities (see section 9.3.E in [26]). One then uses Nadel's vanishing theorem and Green-Lazarsfeld's generic vanishing statement to show that $\text{adj}(\Theta) = \mathcal{O}_A \iff \chi(X, \omega_X) > 0$. Hence, if the adjoint ideal were not trivial, the Euler characteristic of ω_X would be zero, and we would conclude as above by contradicting the fact that Θ must be of general type.*

1.4 Dissertation structure

Having introduced the theory of generic vanishing over the complex numbers and having illustrated its power by sketching the derivation of a few relevant geometric consequences, we can now lay down the structure of this dissertation.

We start in Chapter 2 by recording briefly all the background results we will need: in 2.1 we recall the main definitions and some useful properties of the Fourier-Mukai transform in the context of abelian varieties, in 2.2 we record the relevant definitions of F-singularities; in 2.3 we record results of Pink and Roessler characterizing subvarieties of abelian varieties in positive characteristic; in 2.4 we outline a few useful facts concerning inverse systems that will ease the exposition of the proofs.

As we mentioned at the beginning, until fairly recently, little was known about generic vanishing over fields of positive characteristic. In fact, a theorem of Hacon and Kovács [14] shows that the obvious extension of Green and Lazarsfeld’s generic vanishing theorem is false. Notwithstanding this, recent work of Hacon and Patakfalvi [15] provides a generic vanishing statement in positive characteristic which, albeit necessarily weaker, is strong enough to prove positive characteristic versions of Kawamata’s celebrated characterization of abelian varieties. We start Chapter 2 by summarizing their results, most importantly a positive characteristic version of the implication $(1) \Rightarrow (2)$ from Theorem 2. The rest of Chapter 2 is devoted to generalizing Hacon and Patakfalvi’s theory, by essentially extending Theorem 2. We also provide a number of examples which illustrate the pathologies that one encounters in positive characteristic. The content of this chapter is mainly drawn from the paper [42], which is joint work of the author and Y. Zhang.

Moving on to geometric applications, in Chapter 3 we continue to illustrate the power of this new theory by providing statements and proofs in positive characteristic of the two theorems of Ein and Lazarsfeld that we sketched above.

Finally, in Chapter 4 we explain how one might be able to extend Jiang, Lahoz and Tirabassi’s theorem on the birationality of the tricanonical map to positive characteristic by using ideas in [24] and [43]. We also include some partial results in this direction.

CHAPTER 2

PRELIMINARIES

In this chapter, we record briefly all the background results we need in section 2: in 2.1 we recall the main definitions and some useful properties of the Fourier-Mukai transform in the context of abelian varieties; in 2.2 we record the relevant definitions of F-singularities; in 2.3 we record results of Pink and Roessler characterizing subvarieties of abelian varieties in positive characteristic; in 2.4 we outline a few useful facts concerning inverse systems that will ease the exposition of the proofs and in 2.5 we collect the generic vanishing statements in positive characteristic that will be needed in the sequel.

2.1 Derived categories and Fourier-Mukai transforms

Let A be a g -dimensional abelian variety, denote by $\hat{A} = \text{Pic}^0(A)$ its dual and let $\mathcal{F} \in \text{Coh}(A)$. Let $\mathcal{P} \in \text{Pic}(A \times \hat{A})$ be the Poincaré bundle and denote and consider the usual Fourier-Mukai functors:

$$RS_{A,\hat{A}}^{\mathcal{P}} : D(A) \rightarrow D(\hat{A}), \quad RS_{A,\hat{A}}^{\mathcal{P}}(\bullet) = Rp_{\hat{A}*}(p_A^*(\bullet) \otimes \mathcal{P})$$

even though we will most often omit \mathcal{P} from the notation and simply write $RS_{A,\hat{A}}(\bullet)$.

We start by stating Mukai's inversion theorem in the derived category of quasi-coherent sheaves:

Theorem 10 ([28]). *If $[-g]$ denotes the rightwise shift by g places and -1_A is the inverse on A , the following equalities hold on $D_{qc}(A)$ and $D_{qc}(\hat{A})$*

$$RS_{\hat{A},A} \circ RS_{A,\hat{A}} = (-1_A)^*[-g], \quad RS_{A,\hat{A}} \circ RS_{\hat{A},A} = (-1_{\hat{A}})^*[-g]$$

We will also be using the following two results:

Lemma 11. [28, Proposition 3.8] *The Fourier-Mukai transform commutes with the dualizing functor in $D_{qc}(\hat{A})$ up to inversions and shifts, namely*

$$D_A \circ RS_{\hat{A},A} \simeq \left((-1_A)^* \circ RS_{\hat{A},A} \circ D_{\hat{A}} \right) [g]$$

Lemma 12. [28, Lemma 3.4] Let $\phi : A \rightarrow B$ be an isogeny between abelian varieties and denote by $\hat{\phi} : \hat{B} \rightarrow \hat{A}$ the dual isogeny. Then the following equalities hold on $D_{qc}(B)$ and $D_{qc}(A)$, respectively:

$$\phi^* \circ RS_{\hat{B},B} \simeq RS_{\hat{A},A} \circ \hat{\phi}_*, \quad \phi_* \circ RS_{\hat{A},A} \simeq RS_{\hat{B},B} \circ \hat{\phi}^*.$$

In particular, this holds for the (e -th iterate) Frobenius map F^e and its dual isogeny, namely the Verschiebung map $V^e = \hat{F}^e$.

We will also be using the following simple remark.

Lemma 13. [23, Exercise 5.12] Let $\pi : B \rightarrow A$ be a morphism between abelian varieties and let \mathcal{P} be a locally free sheaf on $A \times \hat{A}$. Denote $\mathcal{P}_\pi = (\pi \times 1_{\hat{A}})^*(\mathcal{P})$. Then

$$S_{\mathcal{P}_\pi} \simeq S_{\mathcal{P}} \circ \pi_*.$$

We next record the notions of homotopy limits and colimits in the derived category. Given a direct system of objects $\mathcal{C}_i \in D(A)$

$$\mathcal{C}_1 \rightarrow \mathcal{C}_2 \rightarrow \dots$$

its homotopy colimit $\underline{\text{hocolim}} \mathcal{C}_i$ is defined by the triangle

$$\oplus \mathcal{C}_i \rightarrow \oplus \mathcal{C}_i \rightarrow \underline{\text{hocolim}} \mathcal{C}_i \xrightarrow{[+1]}$$

where the first map is the homomorphism given by $id - shift$ where $shift : \oplus \mathcal{C}_i \rightarrow \oplus \mathcal{C}_i$ is given on \mathcal{C}_i by the composition $\mathcal{C}_i \rightarrow \mathcal{C}_{i+1} \hookrightarrow \oplus \mathcal{C}_j$.

Given an inverse system of objects $\mathcal{C}_i \in D_{qc}(X)$

$$\mathcal{C}_1 \leftarrow \mathcal{C}_2 \leftarrow \dots$$

its homotopy limit $\underline{\text{holim}} \mathcal{C}_i$ is given by the triangle

$$\underline{\text{holim}} \mathcal{C}_i \rightarrow \prod \mathcal{C}_i \rightarrow \prod \mathcal{C}_i \xrightarrow{+1}$$

where the map between products is $\prod(id - shift)$ and where by product we mean product of chain complexes as opposed to the product inside $D_{qc}(X)$.

Note that if \mathcal{C}_i are coherent sheaves, then $\underline{\text{hocolim}} \mathcal{C}_i = \varinjlim \mathcal{C}_i$ and $\underline{\text{holim}} \mathcal{C}_i = \varprojlim \mathcal{C}_i$.

If X is an n -dimensional variety over a field k and ω_X^\bullet denotes its dualizing complex, so that $\mathcal{H}^{-\dim X}(\omega_X^\bullet) \simeq \omega_X$, we define the dualizing functor D_X on $D_{qc}(X)$ as $D_X(\mathcal{F}) = R\mathcal{H}om(\mathcal{F}, \omega_X^\bullet)$. In this context, Grothendieck duality reads as follows:

Theorem 14. *Let $f : X \rightarrow Y$ be a proper morphism of quasi-projective varieties over a field k . Then for any complex $\mathcal{F} \in D_{qc}(X)$, we have an isomorphism*

$$Rf_* D_X(\mathcal{F}) \simeq D_Y Rf_*(\mathcal{F}).$$

Equivalently, for any $\mathcal{F} \in D_{qc}(X)$ and $\mathcal{E} \in D_{qc}(Y)$, if $\omega_f = \omega_X \otimes f^ \omega_Y$ denotes the relative dualizing sheaf, we have a functorial isomorphism*

$$Rf_* R\mathcal{H}om(\mathcal{F}, Lf^*(\mathcal{E}) \otimes \omega_f[dim X - dim Y]) \simeq R\mathcal{H}om(Rf_* \mathcal{F}, \mathcal{E}).$$

2.2 F-singularities and linear subvarieties of abelian subvarieties

In this section, we recall the basic notions from the theory of F-singularities following [39] and [4]. Let X be a separated scheme of finite type over an F-finite perfect field of characteristic $p > 0$. A variety is a connected reduced equidimensional scheme over k . We denote the canonical sheaf of X by $\omega_X = \mathcal{H}^{-\dim X}(\omega_X^\bullet)$, where $\omega_X^\bullet = \eta^* k$ is the dualizing complex of X and $\eta : X \rightarrow k$ is the structural map. If X is normal, a *canonical divisor* on X is any divisor K_X such that $\omega_X \simeq \mathcal{O}_X(K_X)$.

By a pair (X, Δ) , we mean the combined information of a normal integral scheme X and an effective \mathbb{Q} -divisor Δ . Denote by $F^e : X \rightarrow X$ the e -th iterated absolute Frobenius, where the source has structure map $\eta \circ F^e : X \rightarrow k$. Since $(F^e)^! \omega_X^\bullet = (F^e)^! \eta^! k = \eta^! (F^e)^! k = \eta^! k = \omega_X^\bullet$. In general, for a finite morphism $f : X \rightarrow Y$, a coherent sheaf \mathcal{F} on X and a quasi-coherent sheaf \mathcal{G} on Y , we have the duality $\mathcal{H}om(f_* \mathcal{F}, \mathcal{G}) \simeq f_* \mathcal{H}om(\mathcal{F}, f^! \mathcal{G})$, so the identity $\omega_X^\bullet \rightarrow \omega_X^\bullet \simeq (F^e)^! \omega_X^\bullet$ yields a trace map $F_*^e \omega_X^\bullet \rightarrow \omega_X^\bullet$ and taking cohomology, we obtain $\Phi^e : F_*^e \omega_X \rightarrow \omega_X$. Given a variety X , the *parameter test submodule* $\tau(\omega_X)$ of X is the unique smallest \mathcal{O}_X -submodule $M \subseteq \omega_X$, non-zero on any component of X , such that $\Phi^1(F_* M) \subseteq M$.

Assume that (X, Δ) is a pair such that $K_X + \Delta$ is \mathbb{Q} -Cartier with index not divisible by p . Choose $e > 0$ such that $(p^e - 1)(K_X + \Delta)$ is Cartier and define the line bundle $\mathcal{L}_{e, \Delta} = \mathcal{O}_X((1 - p^e)(K_X + \Delta))$. By [38], there is a canonically determined map $\phi_{e, \Delta} : F_*^e \mathcal{L}_{e, \Delta} \rightarrow \mathcal{O}_X$. We define the *test ideal* $\tau(X, \Delta)$ of the pair (X, Δ) to be the smallest non-zero ideal $J \subseteq \mathcal{O}_X$ such that

$$\phi_{e, \Delta}(F_*^e(J \cdot \mathcal{L}_{e, \Delta})) \subseteq J.$$

Similarly, one defines the *non-F-pure ideal* $\sigma(X, \Delta)$ of (X, Δ) to be the the largest such ideal $J \subseteq \mathcal{O}_X$.

Ever since Hochster and Huneke introduced test ideals and tight closure theory in [22], deep connections have been established between the classes of singularities defined in terms of Frobenius splittings and those arising within the minimal model program. For instance, a normal domain (R, \mathfrak{m}) of characteristic $p > 0$ is said to be F-pure if the inclusion induced by the Frobenius $R \hookrightarrow F_*^e R \equiv R^{1/p^e}$ splits for every e . Similarly, a pair (R, Δ) is said to be F-pure if the inclusion $R \hookrightarrow R^{1/p^e} \hookrightarrow R((p^e - 1)\Delta)^{1/p^e}$ splits for every e and it was shown in [19] that F-pure pairs are the analogues of log canonical pairs in characteristic zero, in the sense that if (X, Δ) is a log canonical pair, then its reduction mod p (X_p, Δ_p) is F-pure for all $p \gg 0$.

In this dissertation, we shall be concerned with the two classes of F-singularities that we define next. We will be recording the original definition in terms of Frobenius splittings and we will then state their description in terms of test ideals that will be used in the sequel.

Definition 15. (i) A pair $(X = \text{Spec } R, \Delta)$ is strongly F-regular if for every non-zero element $c \in R$, there exists e such that the map $R \hookrightarrow R^{1/p^e} \hookrightarrow R((p^e - 1)\Delta)^{1/p^e}$ that sends $1 \mapsto c^{1/p^e} \mapsto c^{1/p^e}$ splits as an R -module homomorphism.

(ii) A reduced connected variety X is F-rational if it is Cohen-Macaulay and there is no non-zero submodule $M \subsetneq \omega_R$ such that the Grothendieck trace map $\Phi_X : F_*^e \omega_X \rightarrow \omega_X$ satisfies $\Phi(F_*^e M) \subseteq M$.

Strongly F-regular pairs are the analog of log terminal pairs in characteristic zero (c.f. [19]) and F-rational varieties are the analogue of varieties with rational singularities (c.f. [41]). The notion of strong F-regularity is also captured by the test ideal, as the following well-known result shows.

Lemma 16. [19, Proposition 2.4] A pair (X, Δ) is strongly F-regular if, and only if, $\tau(X, \Delta) = \mathcal{O}_X$.

Assume that (X, Δ) is a pair, where X is a normal proper variety over an algebraically closed field of characteristic $p > 0$ and $\Delta \geq 0$ is a \mathbb{Q} -divisor such that $K_X + \Delta$ is \mathbb{Q} -Cartier with index not divisible by p . The map $\phi_\Delta^e : F_*^e \mathcal{L}_{e, \Delta} \rightarrow \mathcal{O}_X$ defined in [38] restricts to surjective maps

$$F_*^e(\sigma(X, \Delta) \otimes \mathcal{L}_{e, \Delta}) \longrightarrow \sigma(X, D), \quad F_*^e(\tau(X, \Delta) \otimes \mathcal{L}_{e, \Delta}) \longrightarrow \tau(X, D).$$

The power of vanishing theorems in characteristic zero relies on the fact that they allow us to lift global sections of adjoint bundles. The full space of global sections is not so well

behaved in positive characteristic, so one instead focuses on a subspace of it that is stable under the Frobenius action.

If M is any Cartier divisor, one thus defines the subspace $S^0(X, \tau(X, \Delta) \otimes \mathcal{O}_X(M))$ as

$$\begin{aligned} &:= \bigcap_{n \geq 0} \text{Im} \left(H^0(X, F_*^{ne} \tau(X, \Delta) \otimes \mathcal{L}_{ne, \Delta}(p^{ne} M)) \longrightarrow H^0(X, \tau(X, \Delta) \otimes \mathcal{O}_X(M)) \right) \\ &\subseteq H^0(X, \mathcal{O}_X(M)) \end{aligned}$$

Among the many applications of these subspaces, for instance, they can be used to prove global generation statements: concretely, suppose that X is a d -dimensional variety of characteristic $p > 0$ and that D is a \mathbb{Q} -divisor such that $\mathcal{K}_X + \Delta$ is \mathbb{Q} -Cartier with index not divisible by p . It was shown in [38] that if L and M are Cartier divisors such that $L - K_X - \Delta$ is ample and M is ample and globally generated, then the sheaf $\tau(X, \Delta) \otimes \mathcal{O}_X(L + nM)$ is globally generated for all $n \geq d$ by $S^0(X, \tau(X, \Delta) \otimes \mathcal{O}_X(L + nM))$.

2.3 The Frobenius morphism on abelian varieties

Throughout this paper, A will denote an abelian variety of dimension g over a field k and $\hat{A} = \text{Pic}^0(A)$ will denote the dual abelian variety

Lemma 17. [15, Proposition 2.13] *For a g -dimensional abelian variety A over a field k , the following conditions are equivalent.*

- (i) *There are p^g p -torsion points.*
- (ii) *The Frobenius action $H^g(A, \mathcal{O}_A) \rightarrow H^g(A, \mathcal{O}_A)$ is bijective*
- (iii) *The Frobenius action $H^i(A, \mathcal{O}_A) \rightarrow H^i(A, \mathcal{O}_A)$ is bijective for all $0 \leq i \leq g$*
- (iv) $S^0(A, \omega_A) = H^0(A, \omega_A)$

If any of these equivalent conditions is satisfied, we say that A is *ordinary*. Given an isogeny $\varphi : A \rightarrow B$ between abelian varieties of dimension g , A is ordinary if and only if B is ordinary. Given a surjective morphism $\varphi : A \rightarrow B$ of abelian varieties, if A is ordinary, then so is B (see Lemmas 2.14 and 2.14 in [15]).

We finally record a characterization of linear subvarieties of abelian varieties following [36]. Let A be an abelian variety endowed with an isogeny $\varphi : A \rightarrow A$. We say that A is pure of positive weight if there exist integers $r, s > 0$ such that $\varphi^s = F_{p^r}$ for some model of A over \mathbb{F}_{p^r} . If A is defined over a finite field, we say A is *supersingular* if and only if it is pure

of positive weight for the isogeny given by multiplication by p ; in general, we say that A is supersingular if it is isogenous to a supersingular variety defined over a finite field. We say that A has no supersingular factors if there exists no nontrivial homomorphism to an abelian variety which is pure of positive weight for the isogeny given by multiplication by p . One sees that A has no supersingular factors if there does not exist a nontrivial homomorphism to a supersingular abelian variety. In particular, if A is an ordinary abelian variety, it follows from the observations in the previous paragraph that A has no supersingular factors (see Lemma 2.16 in [15]).

The following result of Pink and Roessler characterizing linear subvarieties of abelian varieties will be crucial in our proof:

Theorem 18. [36, Theorem 2.2] *Let A be an abelian variety over a field K of characteristic $p > 0$ and let $X \subset A$ be a reduced closed subscheme $p(X) \subset X$, where p denotes the isogeny given by multiplication by p . If A has no supersingular factors, then all the maximal dimensional irreducible components of X are completely linear (namely, torsion translates of subabelian varieties).*

2.4 Generalities on inverse systems and spectral sequences

2.4.1 Mittag-Leffler inverse systems

We start by recording a few results that will be used in the sequel, most of which are taken directly from [20]. Recall that a sheaf is countably quasi-coherent if it is quasi-coherent and locally countably generated. Also recall that an inverse system of coherent sheaves $\{\Omega_e\}$ is said to satisfy the Mittag-Leffler condition if for any $e \geq 0$, the image of $\Omega_{e'} \rightarrow \Omega_e$ stabilizes for e' sufficiently large. The inverse limit functor is always left exact in the sense that if

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}_e & \longrightarrow & \mathcal{G}_e & \longrightarrow & \mathcal{H}_e \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{F}_{e-1} & \longrightarrow & \mathcal{G}_{e-1} & \longrightarrow & \mathcal{H}_{e-1} \longrightarrow 0 \end{array}$$

is an exact sequence of inverse systems, then

$$0 \rightarrow \varprojlim \mathcal{F}_e \rightarrow \varprojlim \mathcal{G}_e \rightarrow \varprojlim \mathcal{H}_e$$

is exact in the category of quasi-coherent sheaves. By a theorem of Roos (c.f. Proposition I.4.1 in [20]), the right-derived functors $R^i \varprojlim = 0$ for $i \geq 2$. Hence, we have a long exact sequence

$$0 \rightarrow \varprojlim \mathcal{F}_e \rightarrow \varprojlim \mathcal{G}_e \rightarrow \varprojlim \mathcal{H}_e \rightarrow R^1 \varprojlim \mathcal{F}_e \rightarrow R^1 \varprojlim \mathcal{G}_e \rightarrow R^1 \varprojlim \mathcal{H}_e \rightarrow 0.$$

We start by recording a characterization of the Mittag-Leffler condition in terms of the first right-derived inverse limit.

Lemma 19. *[20, Proposition I.4.9] Let $\{\Omega_e\}_e$ be an inverse system of countably quasi-coherent sheaves on a scheme X of finite type. Then the following conditions are equivalent:*

- (i) $\{\Omega_e\}_e$ satisfies the Mittag-Leffler condition.
- (ii) $R^1 \varprojlim \Omega_e = 0$.
- (iii) $R^1 \varprojlim \Omega_e$ is countably quasi-coherent.

The following is a basic result about the cohomology of an inverse system of sheaves:

Proposition 20. *[20, Theorem I.4.5] Let $\{\Omega_e\}$ be an inverse system of coherent sheaves on a variety X . Let T be a functor on $D(X)$ which commutes with arbitrary direct products. Suppose that $\{\Omega_e\}$ satisfies the Mittag-Leffler condition. Then for each i , there is an exact sequence*

$$0 \rightarrow R^1 \varprojlim R^{i-1} T(\Omega_e) \rightarrow R^i T(\varprojlim \Omega_e) \rightarrow \varprojlim R^i T(\Omega_e) \rightarrow 0.$$

In particular, if for some i , $\{R^{i-1} T(\Omega_e)\}$ satisfies the Mittag-Leffler condition, then by Lemma 7

$$R^i T(\varprojlim \Omega_e) \cong \varprojlim R^i T(\Omega_e).$$

We will be applying this theorem to the push-forward f_* under a proper morphism of schemes. We finally record a standard statement about the commutation of inverse limits and tensor products.

Lemma 21. *Let R be a ring, $I \subset R$ a finitely-generated ideal and $\{M_n\}_n$ a projective system of R -modules of finite length. Then there is an isomorphism*

$$\left(\varprojlim_n M_n \right) \otimes R/I \longrightarrow \varprojlim_n (M_n \otimes R/I).$$

In particular, if $\{\mathcal{F}_e\}$ is an inverse system of coherent sheaves on a scheme X satisfying the Mittag-Leffler condition and $w \in X$ is a point with residue field $k(w) = \mathcal{O}_{X,w}/\mathcal{I}_w$, then there is an isomorphism

$$\left(\varprojlim_e \mathcal{F}_e \right) \otimes k(w) \longrightarrow \left(\varprojlim_e \mathcal{F}_e \otimes k(w) \right)$$

Proof. If $I \subset R$ is finitely generated, then R/I is finitely presented, so it suffices to show that there is an isomorphism

$$\left(\varprojlim_n M_n \right) \otimes N \longrightarrow \varprojlim_n (M_n \otimes N)$$

for any finitely presented R -module. Consider a presentation of N and twist by M_n

$$M_n \otimes R^p \longrightarrow M_n \otimes R^q \longrightarrow M_n \otimes N \longrightarrow 0$$

so $M_n \otimes N = \text{coker} \left(M_n^{\otimes p} \rightarrow M_n^{\otimes q} \right)$. It thus suffices to show that the projective limit of these cokernels is isomorphic to $\text{coker} \left(\varprojlim_n M_n^{\otimes p} \rightarrow \varprojlim_n M_n^{\otimes q} \right)$, but this follows from the fact that a projective system of modules of finite length satisfies the Mittag-Leffler property. \square

Actually a more general statement holds. Recall that a sheaf is countably quasi-coherent if it is quasi-coherent and locally countably generated. Then one has the following:

Lemma 22. *[20, Proposition 4.10] Let $\{\mathcal{F}_e\}_e$ be an inverse system of countably quasi-coherent sheaves on a scheme X of finite type and let E be a flat \mathcal{O}_X -module. Consider the natural map*

$$\alpha : (\varprojlim \mathcal{F}_e) \otimes E \rightarrow \varprojlim (\mathcal{F}_e \otimes E).$$

If $\varprojlim \mathcal{F}_e$ is countably quasi-coherent, then α is injective and if furthermore, $R^1 \varprojlim \Omega_e$ is countably quasi-coherent, then α is surjective.

2.4.2 Inverse systems of convergent spectral sequences

The following observation is taken from [8]. Let $\{E(n)\}$ be an inverse system of spectral sequences with morphisms of spectral sequences $E(n) \rightarrow E(n-1)$ and consider the tri-graded abelian groups $E_{p,q}^r = \varprojlim_n E_{p,q}^r(n)$, with differentials given by the inverse limits of the differentials in the $E(n)$. Concretely, if $d^r(n)$ is the r -th differential in $E(n)$ and $x(n) \in E_{p,q}^r$, then the r -th differential d^r in the limit sequence $E_{p,q}^r$ differential is given by $d^r(x(n)) = d^r(n)(x(n))$. The resulting object is a spectral sequence provided that $H(E_{p,q}^r, d^r) = E_{p,q}^{r+1}$, which is in turn equivalent to showing that

$$H(\varprojlim_n E_{p,q}^r(n), d^r) = \varprojlim_n H(E_{p,q}^r(n), d^r)$$

and this is precisely the statement of Proposition 20 above (for the functor of global sections).

Note that, in particular, if the terms $E_{p,q}^r$ are all finite-dimensional vector spaces, the hypotheses of Proposition 20 hold and all the $\varprojlim^{(1)}$ terms are 0. Besides, given that

the inverse limit of the spectral sequences is again a spectral sequence and provided that every spectral sequence in the inverse system is bounded and convergent, one observes that the limit spectral sequence is also convergent: for fixed p, q , there is a fixed N such that $E_{p,q}^\infty(n) = E_{p,q}^N(n)$.

2.4.3 Morphisms between spectral sequences

We recall the definition of a spectral sequence from EGA III. Let \mathcal{C} be an abelian category. A **(biregular) spectral sequence** E on \mathcal{C} consists of the following ingredients:

1. A family of objects $\{E_r^{p,q}\}$ in \mathcal{C} , where $p, q, r \in \mathbb{Z}$ and $r \geq 2$, such that for any fixed pair (p, q) , $E_r^{p,q}$ stabilizes when r is sufficiently large. We denote the stable objects by $E_\infty^{p,q}$.

2. A family of morphisms $d_r^{p,q} : E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$ satisfying

$$d_r^{p+r, q-r+1} \circ d_r^{p,q} = 0.$$

3. A family of isomorphisms $\alpha_r^{p,q} : \ker(d_r^{p,q}) / \text{Im}(d_r^{p-r, q+r-1}) \xrightarrow{\sim} E_{r+1}^{p,q}$.
4. A family of objects $\{E^n\}$ in \mathcal{C} . For every E^n , there is a bounded decreasing filtration $\{F^p E^n\}$ in the sense that there is some p such that $F^p E^n = E^n$ and there is some p such that $F^p E^n = 0$.
5. A family of isomorphisms $\beta^{p,q} : E_\infty^{p,q} \xrightarrow{\sim} F^p E^{p+q} / F^{p+1} E^{p+q}$.

We say that the spectral sequence $\{E_r^{p,q}\}$ converges to $\{E^n\}$ and write

$$E_2^{p,q} \Rightarrow E^{p+q}.$$

A morphism $\phi : E \rightarrow H$ between two spectral sequences on \mathcal{C} is a family of morphisms $\phi_r^{p,q} : E_r^{p,q} \rightarrow H_r^{p,q}$ and $\phi^n : E^n \rightarrow H^n$ such that ϕ is compatible with d , α , the filtration and β . The following result is useful in order to obtain information about the limiting map ϕ^n from the maps $\phi_2^{p,q}$.

Lemma 23. *Let*

$$\begin{array}{ccc} E_2^{i,j} & \Longrightarrow & E^{i+j} \\ \downarrow \phi_2^{i,j} & & \downarrow \phi^{i+j} \\ H_2^{i,j} & \Longrightarrow & H^{i+j} \end{array}$$

be two spectral sequences with commutative maps. Let l and a be integers. Suppose that $E_2^{i, l-i} = 0$ for $i < a$, $H_2^{i, l-i} = 0$ for $i > a$ and $\phi_2^{a, l-a} = 0$. Then $\phi^l = 0$.

Proof. Since

$$E_3^{i,j} \cong \ker(E_2^{i,j} \rightarrow E_2^{i+2,j-1}) / \text{Im}(E_2^{i-2,j+1} \rightarrow E_2^{i,j}),$$

it follows that $E_3^{i,l-i} = 0$ for $i < a$, $H_3^{i,l-i} = 0$ for $i > a$ and $\phi_3^{a,l-a} = 0$. Hence, $E_\infty^{i,l-i} = 0$ for $i < a$, $H_\infty^{i,l-i} = 0$ for $i > a$ and $\phi_\infty^{a,l-a} = 0$, by induction. Let $\{F^p E^l\}$ and $\{F^p H^l\}$ be the filtration for E^l and H^l , respectively.

We prove by induction that $F^p \phi^l : F^p E^l \rightarrow F^p H^l$ is zero for any p . For $p \geq a+1$, we have that $F^p H^l = 0$. Hence $F^p \phi^l = 0$. Suppose $F^{i+1} \phi^l = 0$ for some $i \leq a$. Since we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & F^{i+1} E^l & \longrightarrow & F^i E^l & \longrightarrow & E_\infty^{i,l-i} \longrightarrow 0 \\ & & \downarrow F^{i+1} \phi^l & & \downarrow F^i \phi^l & & \downarrow \phi_\infty^{i,l-i} \\ 0 & \longrightarrow & F^{i+1} H^l & \longrightarrow & F^i H^l & \longrightarrow & H_\infty^{i,l-i} \longrightarrow 0 \end{array}$$

1. If $i = a$, then $F^{i+1} H^l = 0$. By the Snake Lemma, we have that

$$\text{coker } F^i \phi^l \cong \text{coker } \phi_\infty^{i,l-i} \cong H_\infty^{i,l-i} \cong F^i H^l.$$

Hence, $F^i \phi^l = 0$.

2. If $i < a$, then $E_\infty^{i,l-i} = 0$. By induction, we may assume that $F^{i+1} \phi^l = 0$. By the Snake Lemma, we have that

$$\ker F^i \phi^l \cong \ker F^{i+1} \phi^l \cong F^{i+1} E^l \cong F^i E^l.$$

We also obtain that $F^i \phi^l = 0$.

The lemma follows. \square

We say $\{E^{(m)}\}$ is an inverse system of special sequences if there are morphisms on every entry of every page

$$(\phi^{(m)})_r^{p,q} : (E^{(m+1)})_r^{p,q} \rightarrow (E^{(m)})_r^{p,q}$$

that is compatible with all boundary morphisms $(d^{(m)})_r^{p,q}$. In general, spectral sequences do not behave well under inverse limit. However, we have the following special case:

Theorem 24. *Let $E^{(m)}$ be an inverse system of half-plane special sequences such that all entries are finite-dimensional linear spaces over a field k . Then E^{\lim} is a well-defined special sequence with entries $\varprojlim_m (E^{(m)})_r^{p,q}$. Moreover, the inverse limit of the limits of spectral sequences $(E^{(m)})^n$ coincides with $(E^{\lim})^n$.*

Proof. See [6, Theorem 7.1] and the remark following it. \square

CHAPTER 3

GENERIC VANISHING IN POSITIVE CHARACTERISTIC

A smooth projective variety X over an algebraically closed field is said to have maximal Albanese dimension if it admits a generically finite morphism to an abelian variety $X \rightarrow A$. Over fields of characteristic zero, the main tool that is employed when studying properties of varieties of maximal Albanese dimension is the generic vanishing theorem of Green and Lazarsfeld ([11], [12]). Even though it is shown in [14] that the obvious generalization of this result to fields of positive characteristic is false, recent work of Hacon and Patakfalvi [15] provides a generic vanishing statement in positive characteristic which, albeit necessarily weaker, is strong enough to prove positive characteristic versions of Kawamata's celebrated characterization of abelian varieties.

We start this chapter by summarizing the main results in [15] and we then proceed to generalize them following the approach of Pareschi and Popa in characteristic zero. Excepting the first background section, the content of the chapter is joint work with Y. Zhang and is essentially taken from [42].

3.1 Hacon and Patakfalvi's theory

In this subsection, we collect the results of [15] that we shall be using throughout. The following is the main theorem in [15]:

Theorem 25. [15, Theorem 3.1, Lemma 3.2] *Let A be an abelian variety defined over an algebraically closed field of positive characteristic and let $\Omega_{e+1} \rightarrow \Omega_e$ be an inverse system of coherent sheaves on A .*

1. *If for any sufficiently ample line bundle $L \in \text{Pic}(\hat{A})$ and for any $e \gg 0$ we have $H^i(A, \Omega_e \otimes RS_{\hat{A}, A}(L)^\vee) = 0$ for every $i > 0$, then the complex $\Lambda = \varinjlim RS_{A, \hat{A}}(D_A \Omega_e)$ (which in general is concentrated in degrees $[-g, \dots, 0]$), is actually a quasi-coherent sheaf concentrated in degree 0, namely*

$$\mathcal{H}^i(\Lambda) = \varinjlim \mathcal{H}^i(R\hat{S}(D_A(\Omega_e))) = 0.$$

Besides,

$$\Omega = \varprojlim \Omega_e = \left((-1_A)^* D_A R S_{\hat{A}, A}(\Lambda) \right) [g].$$

2. The condition in 1 is satisfied for coherent Cartier modules: if $F_* \Omega_0 \rightarrow \Omega_0$ is a coherent Cartier module and we denote $\Omega_e = F_*^e \Omega_0$, then for any ample line bundle $L \in \text{Pic}(\hat{A})$, we have

$$H^i(A, \Omega_e \otimes R S_{\hat{A}, A}(L)^\vee \otimes P_\alpha) = 0, \quad \forall e \gg 0, \quad \forall i > 0, \quad \forall \alpha \in \hat{A}.$$

This theorem applies in particular to the Cartier module $F_*^e S^0 a_* \omega_X$, where $a : X \rightarrow A$ is a morphism from a projective variety to an abelian variety. Hacon and Patakfalvi remark in the introduction of [15] that it is expected that

$$\varprojlim F_*^e R^i a_* \omega_X = 0, \quad \forall i > \dim(X/a(X)).$$

We present an approach to this problem in the appendix, which is inspired on an old theorem of Mehta and van der Kallen [27] proving a Grauert-Riemenschneider vanishing theorem for Forbenius split varieties.

From Theorem 25 and the cohomology and base change theorem, one derives the following corollary:

Corollary 26. [15, Corollary 3.5, Corollary 3.6] *With the same notations as above, we have the following:*

- (i) *For every $\alpha \in \hat{A}$ we have $\Lambda \otimes k(\alpha) \simeq \varprojlim H^0(A, \Omega_e \otimes \mathcal{P}_\alpha^\vee)^\vee$, and for every integer $e \geq 0$, $\mathcal{H}^0(\Lambda_e) \otimes k(\alpha) \simeq H^0(A, \Omega_e \otimes \mathcal{P}_\alpha^\vee)^\vee$.*
- (ii) *There exists a proper closed subset $Z \subset \hat{A}$ such that if $i > 0$ and $p^e y \notin Z$ for all $e \gg 0$, then $\varprojlim H^i(A, \Omega_e \otimes \mathcal{P}_\alpha^\vee)^\vee = 0$. Furthermore, if $W^i = \{\alpha \in \hat{A}, \varprojlim H^i(A, \Omega_e \otimes \mathcal{P}_\alpha^\vee)^\vee\}$, then*

$$W^i \subset Z' = \overline{\bigcup_{e \geq 0} \left([p_A^e]^{-1}(Z) \right)_{red}}$$

where $pZ' \subset Z'$. If besides \hat{A} has no supersingular factors, then the top dimensional components of Z' are a finite union of torsion translates of subtori of A .

We finally record two more results from [15] that will provide a simple proof of a special case of our theorem on the singularities of theta divisors. These hold for ordinary abelian varieties and invoke results of Pink and Roessler on linear subvarieties of abelian varieties (c.f. section 2.2).

Proposition 27. [15, Proposition 3.17] *Let A be an ordinary abelian variety and consider the same notations as above. Then each maximal dimensional irreducible component of the set Z of points $\alpha \in \hat{A}$ such that the image of the natural map*

$$\mathcal{H}^0(\Lambda_0) \otimes \mathcal{O}_{\hat{A},\alpha} \longrightarrow \mathcal{H}^0(\Lambda) \otimes \mathcal{O}_{\hat{A},\alpha} \simeq \Lambda \otimes \mathcal{O}_{\hat{A},\alpha}$$

is non-zero, is a torsion translate of an abelian subvariety of \hat{A} and $\Lambda \otimes \mathcal{O}_{\hat{A},\alpha} \neq 0$ if and only if $\mathcal{P}_\alpha^e \in Z$.

Proposition 28. [15, Lemma 3.9, Corollary 3.10] *Let Ω_0 be a coherent sheaf on an abelian variety A and assume that $F_*\Omega_0 \rightarrow \Omega_0$ is surjective. Then $\text{Supp } \Omega = \text{Supp } \Omega_0$, so that $\text{Supp } \Omega$ is a closed subvariety. Let $\hat{B} \subset \hat{A}$ be an abelian subvariety such that*

$$V^0(\Omega_0) = \{\alpha \in \hat{A} : h^0(\Omega_0 \otimes \mathcal{P}_\alpha) \neq 0\}$$

is contained in finitely many translates of \hat{B} . Then $t_x^\Omega \simeq \Omega$ for every $x \in \widehat{\hat{A}/\hat{B}}$. In particular, $\text{Supp } \Omega$ is fibered by the projection $A \rightarrow B$, namely $\text{Supp } \Omega$ is a union of fibers of $A \rightarrow B$.*

Note that, in particular, Proposition 28 applies to the subvariety

$$Z = \left\{ \alpha \in \hat{A} : \text{Im} \left(\mathcal{H}^0(\Lambda_0) \otimes \mathcal{O}_{\hat{A},\alpha} \longrightarrow \Lambda \otimes \mathcal{O}_{\hat{A},\alpha} \right) \neq 0 \right\}$$

from Proposition 27.

3.2 Summary of results

One should regard Theorem 25 as a generalization of (1) \Rightarrow (2) in Theorem 2. It is a natural question to ask what is the appropriate generalization of the statements for (3) and (4) in Theorem 2 to the positive characteristic setting and if all the resulting conditions are equivalent to each other.

In this chapter, we generalize Hacon and Patakfalvi's theorem as follows.

Theorem 29. *Let A be an abelian variety. Let $\{\Omega_e\}$ be an inverse system of coherent sheaves on A satisfying the Mittag-Leffler condition and let $\Omega = \varprojlim \Omega_e$. Let $\Lambda_e = R\hat{S}(D_A(\Omega_e))$ and $\Lambda = \varinjlim \Lambda_e$. The following are equivalent:*

- (1) *For any ample line bundle L on \hat{A} , $H^i(A, \Omega \otimes \hat{L}^\vee) = 0$ for any $i > 0$.*
- (1') *For any fixed positive integer e and any $i > 0$, the homomorphism*

$$H^i(A, \Omega \otimes \hat{L}^\vee) \rightarrow H^i(A, \Omega_e \otimes \hat{L}^\vee)$$

is 0 for any sufficiently ample line bundle L .

(2) $\mathcal{H}^i(\Lambda) = 0$ for any $i \neq 0$.

If any of these conditions is satisfied, then we will call $\{\Omega_e\}$ a GV-inverse system of coherent sheaves.

These conditions imply the following:

(3) For any scheme-theoretic point $P \in A$, if $\text{codim } P < i$, then P is not in the support of

$$\text{Im}(R^i \hat{S}(\Omega) \rightarrow R^i \hat{S}(\Omega_e))$$

for any e .

We also record the following variant of the implication (2) \Rightarrow (3) in the previous theorem.

Proposition 30. Let $\pi : A \rightarrow W$ be a projection between abelian varieties with generic fiber dimension f and with $\dim W = k$. Let $\{\Omega_e\}_e$ be a Cartier module on A and let $S_{A, \hat{W}}$ be the Fourier-Mukai functor with kernel $(\pi \times 1_{\hat{W}})^* \mathcal{P}^{W \times \hat{W}}$. Denote $\Lambda_e = RS_{A, \hat{W}}(D_A(\Omega_e))$. If $P \in \hat{W}$ is a scheme-theoretic point with $\dim(P) > k + f - \ell$, then P is not in the support of the image of the map

$$\varprojlim_e R^\ell S_{A, \hat{W}}(\Omega_e) \rightarrow R^\ell S_{A, \hat{W}}(\Omega_e).$$

Moreover, if the inverse system $\{R^\ell S_{A, \hat{W}}(\Omega_e)\}_e$ satisfies the Mittag-Leffler condition, then the support of the image of the above map is closed and its codimension is $\geq \ell - f$.

Theorem 31 (c.f. Theorem 41). In the setting of Theorem 29, if we further assume that $\{R^i \hat{S}(\Omega_e)\}$ satisfies the Mittag-Leffler condition for any $i \geq 0$, then the following condition implies (1), (1') and (2).

(3') For any scheme-theoretic point $P \in A$, if $\text{codim } P < i$, then P is not in the support of

$$\text{Im}(\varprojlim R^i \hat{S}(\Omega_e) \rightarrow R^i \hat{S}(\Omega_e))$$

for any e .

We should remark that even if $\{\Omega_e\}$ is a Cartier module, $\{R^i \hat{S}(\Omega_e)\}$ does not necessarily satisfy the Mittag-Leffler condition (see Example 36). We do not know whether (3) implies (3'). The statement about $V^i(\Omega)$ is still missing. We will give an example (see Example 37) where the chain of inclusions for $V^i(\Omega)$ fails. Since the support of $\text{Im}(R^i \hat{S}(\Omega) \rightarrow R^i \hat{S}(\Omega_e))$ is usually not closed, it does not make sense to consider its codimension as in Theorem 2 (c.f. Lemma 40).

Recall that in a sequence of papers [32, 31, 34], Pareschi and Popa introduced a notion of M-regularity which parallels and strengthens the usual Castelnuovo-Mumford regularity with respect to polarizations on abelian varieties and developed several results on global generation. In [34], the following characterization of M-regularity is given.

Theorem 32. *Let A be an abelian variety and \mathcal{F} be a coherent sheaf on A satisfying the Generic Vanishing conditions. The following conditions are equivalent:*

1. \mathcal{F} is M-regular, namely $R^0\hat{S}(D_A(\mathcal{F}))$ is torsion-free.
2. $\text{codim Supp } R^i\hat{S}(\mathcal{F}) > i$ for any $i > 0$
3. $\text{codim } V^i(\mathcal{F}) > i$ for any $i > 0$.

We will generalize the theorem above to inverse systems as follows.

Theorem 33 (c.f. Theorem 39). *Let A be an abelian variety and $\{\Omega_e\}$ be a GV-inverse system of coherent sheaves on A such that*

1. $\{\Omega_e\}$ is M-regular in the sense that $\mathcal{H}^0(\Lambda)$ is torsion-free.

Then

- (2) *for any $i > 0$ and any scheme-theoretic point $P \in A$, if $\text{codim } P \leq i$, then P is not in the support of*

$$\text{Im}(R^i\hat{S}(\Omega) \rightarrow R^i\hat{S}(\Omega_e))$$

for any e .

3.3 Examples

We next present several examples illustrating the pathologies that appear in the context of Cartier modules.

Example 34. *This example first appeared in [15, Example 3.4.4]. Let A be an elliptic curve, $\Omega_0 = \omega_A$, $\Omega_e = F_*^e \Omega_0$ and $\alpha : F_* \Omega_0 \rightarrow \Omega_0$ be the trace map.*

When A is ordinary, then $\Lambda = \varinjlim R\hat{S}(D_A(\Omega_e)) = \bigoplus_{y \in A_p} k(y)$ where A_p denotes the set of all p^∞ -torsion points in \hat{A} . By [15, Proposition 3.4.1], $V^1(\Omega) = A_p$ which is a countable dense set in \hat{A} .

However, suppose \mathcal{F} is a coherent sheaf satisfying the Generic Vanishing conditions, then $V^1(\mathcal{F})$ is a closed subset of dimension 0 or empty by Theorem 2.

We should notice that in this example, the support of the image of $R^1\hat{S}(\Omega) \rightarrow R^1\hat{S}(\Omega_e)$ is the set of p^e -torsion points which is closed for any $e > 0$. \square

It should be noticed that in the previous example, $\{R^i\hat{S}(\Omega_e)\}$ satisfies the Mittag-Leffler condition for any i . We will see that this is not valid in general in the following examples.

Example 35. Let A be an elliptic curve. Let $\hat{0} \in \hat{A}$ correspond to the trivial line bundle on A . Let $W_e = \mathcal{O}_{\hat{A}}(-e \cdot \hat{0})$ and $\psi_e : W_{e+1} \rightarrow W_e$ be the inclusion. Clearly, the inverse system of coherent sheaves $\{W_e\}$ does not satisfy the Mittag-Leffler condition. Since the W_e are antiample, $R^0S(W_e) = 0$. Let $\Omega_e = R^1S(W_e) = RS(W_e)[1]$. Notice that we have short exact sequences

$$0 \rightarrow W_{e+1} \rightarrow W_e \rightarrow k(\hat{0}) \rightarrow 0.$$

The Fourier-Mukai transform induces the following long exact sequences

$$\begin{array}{ccccccccc} R^0S(W_e) & \longrightarrow & R^0S(k(\hat{0})) & \longrightarrow & R^1S(W_{e+1}) & \longrightarrow & R^1S(W_e) & \longrightarrow & R^1S(k(\hat{0})) \\ \parallel & & \parallel & & \parallel & & \parallel & & \parallel \\ 0 & & \mathcal{O}_A & & \Omega_{e+1} & & \Omega_e & & 0. \end{array}$$

In particular, $\Omega_{e+1} \rightarrow \Omega_e$ is surjective. Thus, $\{\Omega_e\}$ satisfies the Mittag-Leffler condition. On the other hand, $R^0\hat{S}(\Omega_e) = (-1_{\hat{A}})^*W_e$ does not satisfy the Mittag-Leffler condition.

We claim that $\{\Omega_e\}$ above is a GV-inverse system of coherent sheaves. Indeed,

$$R\hat{S}(D_A(\Omega_e)) = (-1_{\hat{A}})^*D_{\hat{A}}(R\hat{S}(\Omega_e))[-1] = D_{\hat{A}}(W_e)[-1] = \mathcal{O}_{\hat{A}}(e \cdot \hat{0}).$$

In particular, $R^{-1}\hat{S}(D_A(\Omega_e)) = 0$. Taking the direct limit, we conclude that $\mathcal{H}^{-1}(\Lambda) = 0$.

We calculate the cohomology support loci. Let $\alpha \in \hat{A}$ and $P_\alpha \in \text{Pic}^0(A)$ be the corresponding topologically trivial line bundle. We have

$$\begin{aligned} H^i(A, \Omega_e \otimes P_\alpha) &\cong H^i(A, R^1S(W_e) \otimes P_\alpha) \\ &\cong H^{i+1}(A, RS(W_e) \otimes P_\alpha) \\ &\cong H^{i+1}(\hat{A}, W_e \otimes R\hat{S}(P_\alpha)) \\ &\cong H^i(\hat{A}, W_e \otimes R^1\hat{S}(P_\alpha)) \\ &\cong H^i(\hat{A}, W_e \otimes k(-\alpha)), \end{aligned}$$

Hence, $H^0(A, \Omega_e \otimes P_\alpha) \cong k$ and $H^1(A, \Omega_e \otimes P_\alpha) = 0$. Taking the inverse limit, we have $H^i(A, \Omega \otimes P_\alpha) = \varprojlim H^i(A, \Omega_e \otimes P_\alpha) = k(-\alpha)$ if $i = 0$ and $\alpha \neq \hat{0}$, and $H^i(A, \Omega \otimes P_\alpha) = 0$ otherwise. We conclude that

$$V^1(\Omega) = \emptyset$$

and

$$V^0(\Omega) = \hat{A} - \{\hat{0}\}.$$

In particular,

$$V^0(\Omega) = \text{Supp } R^0\hat{S}(\Omega) = \text{Supp } (\text{Im} R^0\hat{S}(\Omega) \rightarrow R^0\hat{S}(\Omega_e)) = \hat{A} - \{\hat{0}\}$$

are not countable unions of closed subsets.

We should notice that $V^1(\Omega) \not\supseteq \text{Supp } R^1\hat{S}(\Omega)$. If \mathcal{F} is a coherent sheaf, then it is a consequence of cohomology and base change that $V^i(\mathcal{F}) \supseteq \text{Supp } R^i\hat{S}(\mathcal{F})$. \square

We can easily modify Example 35 to obtain a Cartier module.

Example 36. Let A be a supersingular elliptic curve. Let $\{W_e\}$ and $\{\Omega_e\}$ be the same as in Example 35. Let us consider the inverse system $\{\Omega_{p^e}\}$. Notice that since A is supersingular,

$$W_{p^e} = \mathcal{O}_{\hat{A}}(-p^e \cdot \hat{0}) = V^{*,e}(W_1),$$

where the Verschiebung $V : \hat{A} \rightarrow \hat{A}$ is the dual of the Frobenius. We have

$$\Omega_{p^e} \cong RS(W_{p^e})[1] \cong RS(V^{*,e}(W_1))[1] \cong F_*^e(RS(W_1))[1] \cong F_*^e\Omega_1,$$

where the third isomorphism is by Lemma 12. Thus, $\{\Omega_{p^e}\}$ is a Cartier module. The calculation of inverse limits remains unchanged as in Example 35. In particular, $V^0(\Omega) = \hat{A} - \{\hat{0}\}$ is not a countable union of closed subvarieties. This gives a negative answer to [15, Question 3.4.3] in the supersingular case. \square

The following example shows that the chain of inclusions fails for GV-inverse system of coherent sheaves.

Example 37. Let A be an elliptic curve. Let $\Omega_0 = \mathcal{O}_A$ and Ω_{e+1} be the nonsplitting extension of \mathcal{O}_A and Ω_e ,

$$0 \rightarrow \mathcal{O}_A \rightarrow \Omega_{e+1} \rightarrow \Omega_e \rightarrow 0.$$

Then $H^0(A, \Omega_e) \cong H^1(A, \Omega_e) \cong k$ for any $e \geq 0$. Since $\Omega_{e+1} \rightarrow \Omega_e$ is surjective, the inverse system $\{\Omega_e\}$ satisfies the Mittag-Leffler condition. It is easy to check that

$$R^{-1}\hat{S}(D_A(\Omega_e)) = 0$$

for any $e \geq 0$ by induction. Hence, $\{\Omega_e\}$ is a GV-inverse system.

We now compute the cohomology support loci $V^0(\Omega)$ and $V^1(\Omega)$. Suppose $P_\alpha \in \text{Pic}^0(A)$ and $P_\alpha \neq \mathcal{O}_A$. By the long exact sequence,

$$\begin{aligned} 0 \rightarrow H^0(A, P_\alpha) \rightarrow H^0(A, \Omega_{e+1} \otimes P_\alpha) \rightarrow H^0(A, \Omega_e \otimes P_\alpha) \\ \rightarrow H^1(A, P_\alpha) \rightarrow H^1(A, \Omega_{e+1} \otimes P_\alpha) \rightarrow H^1(A, \Omega_e \otimes P_\alpha) \rightarrow 0, \end{aligned}$$

we have that

$$H^0(A, \Omega_e \otimes P_\alpha) = H^1(A, \Omega_e \otimes P_\alpha) = 0$$

for any $e \geq 0$. Thus, we only need to consider whether $\hat{0} \in \hat{A}$ is in the cohomology support loci of Ω .

We have the following exact sequence,

$$\begin{aligned} 0 \rightarrow H^0(A, \mathcal{O}_A) \rightarrow H^0(A, \Omega_{e+1}) \rightarrow H^0(A, \Omega_e) \\ \xrightarrow{\cong} H^1(A, \mathcal{O}_A) \rightarrow H^1(A, \Omega_{e+1}) \rightarrow H^1(A, \Omega_e) \rightarrow 0, \end{aligned}$$

where the isomorphism is by the assumption that the extension of \mathcal{O}_A and Ω_e is nonsplitting. Hence, $H^0(A, \Omega_{e+1}) \rightarrow H^0(A, \Omega_e)$ is zero and $H^1(A, \Omega_{e+1}) \rightarrow H^1(A, \Omega_e)$ is an isomorphism for any $e \geq 0$. Taking the inverse limit, we obtain that

$$H^0(A, \Omega) = H^0(A, \varprojlim \Omega_e) \cong \varprojlim H^0(A, \Omega_e) = 0,$$

and

$$H^1(A, \Omega) = H^1(A, \varprojlim \Omega_e) \cong \varprojlim H^1(A, \Omega_e) \cong H^1(A, \Omega_0) \cong k.$$

Thus, $V^0(\Omega) = \emptyset$ and $V^1(\Omega) = \{\hat{0}\}$. The chain of inclusions fails.

When A is supersingular, by [18, Lemma 4.12], we have that $F_*^e \omega_A \cong \Omega_{p^e-1}$. The nontrivial map $\Omega_{p^e-1} \rightarrow \Omega_0$ induces $F_*^e \omega_A \rightarrow \omega_A$, which is isomorphic to the trace map up to a scale. Hence, the inverse system $\{\Omega_{p^e-1}\}$ is a Cartier module and is the same as Example 34. \square

3.4 Main theorem

We are now in a position to embark on the proof of Theorem 29.

3.4.1 WIT versus limit of Kodaira vanishing

Theorem 38. *Let A be an abelian variety of dimension g . Let $\{\Omega_e\}$ be an inverse system of coherent sheaves on A satisfying the Mittag-Leffler condition and let $\Omega = \varprojlim \Omega_e$. Let $\Lambda_e = R\hat{S}(D_A(\Omega_e))$ and $\Lambda = \varinjlim \Lambda_e$. The following are equivalent:*

1. For any ample line bundle L on \hat{A} , $H^i(A, \Omega \otimes \hat{L}^\vee) = 0$ for any $i > 0$.
2. For any nonnegative integer e and any ample line bundle L on \hat{A} , there exists an integer $m(e, L)$ such that for any $m \geq m(e, L)$, the natural map

$$H^i(A, \Omega \otimes \widehat{mL}^\vee) \rightarrow H^i(A, \Omega_e \otimes \widehat{mL}^\vee)$$

is zero for any $i > 0$.

3. $\mathcal{H}^i(\Lambda) = 0$ for $i \neq 0$.

Proof. (1) \Rightarrow (2). Obvious.

(2) \Rightarrow (3). By construction, we have $\mathcal{H}^i(\Lambda) = 0$ when $i < -g$ or $i > 0$. Thus, we may pick $j < 0$ as the smallest integer such that $\mathcal{H}^j(\Lambda) \neq 0$. Since $\mathcal{H}^j(\Lambda) = \varinjlim \mathcal{H}^j(\Lambda_e)$, we may fix $e > 0$ such that the image of $\mathcal{H}^j(\Lambda_e) \rightarrow \mathcal{H}^j(\Lambda)$ is non-zero. Let L be a sufficiently large multiple of a fixed ample line bundle on \hat{A} such that

- (i) $\mathcal{H}^j(\Lambda_e) \otimes L$ is globally generated,
- (ii) $H^i(\hat{A}, \mathcal{H}^l(\Lambda_e) \otimes L) = 0$ for $i > 0$ and $l \in [-g, 0]$, and
- (iii) $H^i(A, \Omega \otimes \hat{L}^\vee) \rightarrow H^i(A, \Omega_e \otimes \hat{L}^\vee)$ is zero for any $i \neq 0$.

Notice that (i) and (ii) can be achieved by Serre Vanishing and (iii) can be achieved by the hypothesis in condition (2).

Using Grothendieck's spectral sequence, we have

$$\begin{array}{ccc} E_{2,e}^{i,l} = R^i\Gamma(\mathcal{H}^l(\Lambda_e) \otimes L) & \Longrightarrow & R^{i+l}\Gamma(\Lambda_e \otimes L) \\ \downarrow & & \downarrow \\ E_2^{i,l} = R^i\Gamma(\mathcal{H}^l(\Lambda) \otimes L) & \Longrightarrow & R^{i+l}\Gamma(\Lambda \otimes L), \end{array}$$

where the vertical arrows are compatible by the functoriality of the spectral sequence. By our choice of j , we have that $E_2^{i,l} = 0$ for all $l < j$. By (ii), $E_{2,e}^{i,l} = 0$ for any $i \neq 0$. Hence, the spectral sequence degenerates to the following commutative diagram:

$$\begin{array}{ccc} R^0\Gamma(\mathcal{H}^j(\Lambda_e) \otimes L) & \xrightarrow{\simeq} & R^j\Gamma(\Lambda_e \otimes L) \\ \downarrow & & \downarrow \\ R^0\Gamma(\mathcal{H}^j(\Lambda) \otimes L) & \xrightarrow{\simeq} & R^j\Gamma(\Lambda \otimes L). \end{array}$$

By (i), the image of $R^0\Gamma(\mathcal{H}^j(\Lambda_e) \otimes L) \rightarrow R^0\Gamma(\mathcal{H}^j(\Lambda) \otimes L)$ is non-zero. Hence, the image of $R^j\Gamma(\Lambda_e \otimes L) \rightarrow R^j\Gamma(\Lambda \otimes L)$ is non-zero.

On the other hand, we have

$$\begin{aligned}
D_k(R^j\Gamma(\Lambda \otimes L)) &\cong D_k(\varinjlim R^j\Gamma(\Lambda_e \otimes L)) \\
&\cong \varprojlim D_k R^j\Gamma(\Lambda_e \otimes L) \\
&\cong \varprojlim D_k R^j\Gamma(R\hat{S}(D_A(\Omega_e)) \otimes L) \\
&\cong \varprojlim D_k R^j\Gamma(D_A(\Omega_e \otimes \hat{L}^\vee)) \\
&\cong \varprojlim R^{-j}\Gamma(D_A(D_A(\Omega_e \otimes \hat{L}^\vee))) \\
&\cong \varprojlim R^{-j}\Gamma(\Omega_e \otimes \hat{L}^\vee),
\end{aligned}$$

and similarly, $D_k(R^j\Gamma(\Lambda_e \otimes L)) \cong R^{-j}\Gamma(\Omega_e \otimes \hat{L}^\vee)$. Since the inverse system $\{\Omega_e\}$ satisfies the Mittag-Leffler condition, we have

$$H^i(A, \Omega \otimes \hat{L}^\vee) = H^i(A, \varprojlim \Omega_e \otimes \hat{L}^\vee) \cong \varprojlim H^i(A, \Omega_e \otimes \hat{L}^\vee)$$

for any i . Thus by (iii), $\varprojlim R^{-j}\Gamma(\Omega_e \otimes \hat{L}^\vee) \rightarrow R^{-j}\Gamma(\Omega_e \otimes \hat{L}^\vee)$ is zero. Hence, $D_k(R^j\Gamma(\Lambda \otimes L)) \rightarrow D_k(R^j\Gamma(\Lambda_e \otimes L))$ is zero, a contradiction.

(3) \Rightarrow (1). Recall that we have the following spectral sequence,

$$H^i(\hat{A}, \mathcal{H}^l(\Lambda) \otimes L) \Rightarrow R^{i+l}\Gamma(\hat{A}, \Lambda \otimes L).$$

Since $\mathcal{H}^i(\Lambda) = 0$ for any $i \neq 0$, the spectral sequence degenerates to

$$H^i(\hat{A}, \mathcal{H}^0(\Lambda) \otimes L) \cong R^i\Gamma(\hat{A}, \Lambda \otimes L).$$

If $i > 0$, then by the isomorphism shown in the previous step,

$$H^i(A, \Omega \otimes \hat{L}^\vee) \cong D_k(R^{-i}\Gamma(\hat{A}, \Lambda \otimes L)) \cong D_k(H^{-i}(\hat{A}, \mathcal{H}^0(\Lambda) \otimes L)) = 0.$$

□

3.4.2 WIT versus the supports of $R^i\hat{S}(\Omega)$

Theorem 39. *Let $\{\Omega_e\}$ be an inverse system of coherent sheaves on a g -dimensional abelian variety satisfying the Mittag-Leffler condition and let $\Omega = \varprojlim \Omega_e$. Let $\Lambda_e = R\hat{S}(D_A(\Omega_e))$ and $\Lambda = \varinjlim \Lambda_e$. If $\mathcal{H}^j(\Lambda) = 0$ for any $j \neq 0$, then for any scheme-theoretic point P with $\text{codim } P < i$, we have*

$$P \notin \text{Supp} (Im(R^i\hat{S}(\Omega) \rightarrow R^i\hat{S}(\Omega_e)))$$

for any $e \geq 0$. Moreover, if $\mathcal{H}^0(\Lambda)$ is torsion-free, then for any $i > 0$ and any scheme-theoretic point P with $\text{codim } P \leq i$, we have

$$P \notin \text{Supp} (Im(R^i\hat{S}(\Omega) \rightarrow R^i\hat{S}(\Omega_e)))$$

for any $e \geq 0$.

Proof. Fix a scheme-theoretic point $P \in \hat{A}$ such that $\dim P = d$. Since localization at P is exact, we have the following commutative diagram of spectral sequences:

$$\begin{array}{ccc} \mathcal{E}xt^i(\mathcal{H}^j(\Lambda), \mathcal{O}_{\hat{A}})_P & \Longrightarrow & \mathcal{E}xt^{i-j}(\Lambda, \mathcal{O}_{\hat{A}})_P \\ \downarrow \phi_2^{i,-j} & & \downarrow \phi^{i-j} \\ \mathcal{E}xt^i(\mathcal{H}^j(\Lambda_e), \mathcal{O}_{\hat{A}})_P & \Longrightarrow & \mathcal{E}xt^{i-j}(\Lambda_e, \mathcal{O}_{\hat{A}})_P. \end{array}$$

Since $\dim P = d$, by [21, III.6.8 and III.6.10A],

$$\mathcal{E}xt^i(\mathcal{H}^j(\Lambda_e), \mathcal{O}_{\hat{A}})_P \cong Ext_{\mathcal{O}_{\hat{A},P}}^i(\mathcal{H}^j(\Lambda_e)_P, \mathcal{O}_{\hat{A},P}) = 0,$$

when $i > g - d$. Let $l = i - j > g - d$ and $a = l - 1$. When $i \leq a$, we have $j = i - l < 0$, hence $\mathcal{E}xt^i(\mathcal{H}^j(\Lambda), \mathcal{O}_{\hat{A}})_P = 0$. When $i > a$, we have $i \geq l > g - d$, hence $\mathcal{E}xt^i(\mathcal{H}^j(\Lambda_e), \mathcal{O}_{\hat{A}})_P = 0$. We may apply Lemma 23 and obtain that the natural map $\mathcal{E}xt^l(\Lambda, \mathcal{O}_{\hat{A}})_P \rightarrow \mathcal{E}xt^l(\Lambda_e, \mathcal{O}_{\hat{A}})_P$ is zero when $l > g - d$.

It is easy to see that

$$\begin{aligned} cl\mathcal{E}xt^l(\Lambda_e, \mathcal{O}_{\hat{A}}) &\cong \mathcal{H}^{l-g}(D_{\hat{A}}(\Lambda_e)) \cong \mathcal{H}^{l-g}(D_{\hat{A}}(R\hat{S}(D_A(\Omega_e)))) \\ &\cong \mathcal{H}^{l-g}((-1_{\hat{A}})^* R\hat{S}(D_A(D_A(\Omega_e)))[g]) \cong \mathcal{H}^l((-1_{\hat{A}})^* R\hat{S}(\Omega_e)) \\ &\cong (-1_{\hat{A}})^* R^l \hat{S}(\Omega_e) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \mathcal{E}xt^l(\Lambda, \mathcal{O}_{\hat{A}}) &\cong \mathcal{H}^{l-g}(D_{\hat{A}}(\Lambda)) \cong \mathcal{H}^{l-g}(D_{\hat{A}}(\varinjlim R\hat{S}(D_A(\Omega_e)))) \\ &\cong \mathcal{H}^{l-g}(\varprojlim D_{\hat{A}}(R\hat{S}(D_A(\Omega_e)))) \cong \mathcal{H}^l(\varprojlim (-1_{\hat{A}})^* R\hat{S}(\Omega_e)) \\ &\cong (-1_{\hat{A}})^* \mathcal{H}^l(\varprojlim R\hat{S}(\Omega_e)) \end{aligned} \tag{3.2}$$

Then for any d -dimensional point P ,

$$\mathcal{H}^l(\varprojlim R\hat{S}(\Omega_e))_P \rightarrow R^l \hat{S}(\Omega_e)_P$$

is zero for any $l > g - d$. Notice that $R^l \hat{S}(\Omega)_P \rightarrow R^l \hat{S}(\Omega_e)_P$ factors as

$$R^l \hat{S}(\Omega)_P = \mathcal{H}^l(R\hat{S}(\varinjlim \Omega_e))_P \rightarrow \mathcal{H}^l(\varprojlim R\hat{S}(\Omega_e))_P \rightarrow \mathcal{H}^l(R\hat{S}(\Omega_e))_P = R^l \hat{S}(\Omega_e)_P.$$

We conclude that $R^l \hat{S}(\Omega)_P \rightarrow R^l \hat{S}(\Omega_e)_P$ is zero when $l > g - \dim P$. The first part of the proposition follows from the exactness of localization at P .

If $\mathcal{H}^0(\Lambda)$ is torsion-free, we only need to check the case that $l = i - j = g - d$. Let $a = l$. When $i < a$, we have that $j = i - l < 0$, hence $\mathcal{E}xt^i(\mathcal{H}^j(\Lambda), \mathcal{O}_{\hat{A}})_P = 0$. When $i > a$, we have $i > l = g - d$, hence $\mathcal{E}xt^i(\mathcal{H}^j(\Lambda_e), \mathcal{O}_{\hat{A}})_P = 0$. In order to apply Lemma 23, we only need to check that

$$\mathcal{E}xt^{g-d}(\mathcal{H}^0(\Lambda), \mathcal{O}_{\hat{A}})_P \rightarrow \mathcal{E}xt^{g-d}(\mathcal{H}^0(\Lambda_e), \mathcal{O}_{\hat{A}})_P$$

is zero. Let \mathcal{T} be the torsion part of $\mathcal{H}^0(\Lambda_e)$ and $\mathcal{F} \cong \mathcal{H}^0(\Lambda_e)/\mathcal{T}$. We have the following exact sequence,

$$\mathcal{E}xt^{g-d}(\mathcal{F}, \mathcal{O}_{\hat{A}})_P \rightarrow \mathcal{E}xt^{g-d}(\mathcal{H}^0(\Lambda_e), \mathcal{O}_{\hat{A}})_P \rightarrow \mathcal{E}xt^{g-d}(\mathcal{T}, \mathcal{O}_{\hat{A}})_P \rightarrow \mathcal{E}xt^{g-d+1}(\mathcal{F}, \mathcal{O}_{\hat{A}})_P.$$

Since \mathcal{F} is torsion-free, by the argument in [34, Lemma 2.9], we have that when $l = g - d > 0$,

$$\mathcal{E}xt^{g-d}(\mathcal{F}, \mathcal{O}_{\hat{A}})_P = \mathcal{E}xt^{g-d+1}(\mathcal{F}, \mathcal{O}_{\hat{A}})_P = 0.$$

Hence,

$$\mathcal{E}xt^{g-d}(\mathcal{H}^0(\Lambda_e), \mathcal{O}_{\hat{A}})_P \cong \mathcal{E}xt^{g-d}(\mathcal{T}, \mathcal{O}_{\hat{A}})_P.$$

We only need to show that

$$\mathcal{E}xt^{g-d}(\mathcal{H}^0(\Lambda), \mathcal{O}_{\hat{A}})_P \rightarrow \mathcal{E}xt^{g-d}(\mathcal{T}, \mathcal{O}_{\hat{A}})_P$$

is zero. Notice that this map is induced by $\mathcal{T} \rightarrow \mathcal{H}^0(\Lambda_e) \rightarrow \mathcal{H}^0(\Lambda)$ where \mathcal{T} is a torsion sheaf and $\mathcal{H}^0(\Lambda)$ is torsion-free. Thus, $\mathcal{T} \rightarrow \mathcal{H}^0(\Lambda)$ is zero. Hence,

$$\mathcal{E}xt^{g-d}(\mathcal{H}^0(\Lambda), \mathcal{O}_{\hat{A}})_P \rightarrow \mathcal{E}xt^{g-d}(\mathcal{T}, \mathcal{O}_{\hat{A}})_P$$

is zero. □

3.4.3 The case when $\{R^i \hat{S}(\Omega_e)\}$ satisfies the Mittag-Leffler condition

In this section, we will consider the Mittag-Leffler condition on the Fourier-Mukai transform of the inverse system $\{\Omega_e\}$. We are able to recover Theorem 2 fully in this setting. However, we remind the reader that even when $\{\Omega_e\}$ is a Cartier module, the inverse system $\{R^i \hat{S}(\Omega_e)\}$ does not necessarily satisfy the Mittag-Leffler condition (see Example 36).

Proposition 40. *For any $0 \leq i \leq g$, if $\{R^i \hat{S}(\Omega_e)\}$ satisfies the Mittag-Leffler condition, then the support of $\text{Im} \left(\varprojlim R^i \hat{S}(\Omega_e) \rightarrow R^i \hat{S}(\Omega_e) \right)$ is closed for any $e \geq 0$.*

Proof. Since we assume $\{R^i \hat{S}(\Omega_e)\}$ satisfies the Mittag-Leffler condition, the image of

$$R^i \hat{S}(\Omega_d) \rightarrow R^i \hat{S}(\Omega_e)$$

stabilizes when d is sufficiently large. The stable image coincides with

$$\mathrm{Im}(\varprojlim R^i \hat{S}(\Omega_e) \rightarrow R^i \hat{S}(\Omega_e)).$$

Since $R^i \hat{S}(\Omega_d)$ and $R^i \hat{S}(\Omega_e)$ are both coherent, the proposition follows. \square

By the proposition above, it makes sense to talk about the codimension of the support of $\mathrm{Im}(\varprojlim R^i \hat{S}(\Omega_e) \rightarrow R^i \hat{S}(\Omega_e))$.

We are able to recover the missing implication in Theorem 29.

Theorem 41. *Let A be an abelian variety. Let $\{\Omega_e\}$ be an inverse system of coherent sheaves on A satisfying the Mittag-Leffler condition and let $\Omega = \varprojlim \Omega_e$. Let $\Lambda_e = R\hat{S}(D_A(\Omega_e))$ and $\Lambda = \varinjlim \Lambda_e$. Suppose that the inverse system $\{R^i \hat{S}(\Omega_e)\}$ satisfies the Mittag-Leffler condition for all i and*

$$\mathrm{codim} \mathrm{Supp} \left(\mathrm{Im} \left(\varprojlim R^i \hat{S}(\Omega_e) \rightarrow R^i \hat{S}(\Omega_e) \right) \right) \geq i,$$

for any $0 \leq i \leq g$ and e sufficiently large. Then for any ample line bundle L on \hat{A} , we have $H^i(A, \Omega \otimes \hat{L}^\vee) = 0$ for any $i > 0$.

Proof. To simplify our notation, we denote

$$\mathrm{Im}_{i,e} = \mathrm{Im} \left(\varprojlim R^i \hat{S}(\Omega_e) \rightarrow R^i \hat{S}(\Omega_e) \right).$$

Let p and q be two non-negative integers satisfying $p + q > g$ and L be any ample line bundle on \hat{A} . The map

$$H^p(\hat{A}, \varprojlim R^q \hat{S}(\Omega_e) \otimes L^\vee) \rightarrow H^p(\hat{A}, R^q \hat{S}(\Omega_e) \otimes L^\vee)$$

factors through $H^p(\hat{A}, \mathrm{Im}_{q,e} \otimes L^\vee)$. Since we assume that

$$\mathrm{codim} \mathrm{Supp} (\mathrm{Im}_{q,e}) \geq q > g - p,$$

the cohomology

$$H^p(\hat{A}, \mathrm{Im}_{q,e} \otimes L^\vee) = 0.$$

Thus,

$$H^p(\hat{A}, \varprojlim R^q \hat{S}(\Omega_e) \otimes L^\vee) \rightarrow H^p(\hat{A}, R^q \hat{S}(\Omega_e) \otimes L^\vee)$$

is the zero map. By the Mittag-Leffler condition of $\{R^q\hat{S}(\Omega_e)\}$, we have the following isomorphism

$$H^p(\hat{A}, \varprojlim R^q\hat{S}(\Omega_e) \otimes L^\vee) \cong \varprojlim H^p(\hat{A}, R^q\hat{S}(\Omega_e) \otimes L^\vee).$$

whence the natural maps

$$\varprojlim H^p(\hat{A}, R^q\hat{S}(\Omega_e) \otimes L^\vee) \rightarrow H^p(\hat{A}, R^q\hat{S}(\Omega_e) \otimes L^\vee)$$

are all zero for any e sufficiently large. By the universal property of inverse limits, we conclude that

$$H^p(\hat{A}, \varprojlim R^q\hat{S}(\Omega_e) \otimes L^\vee) \cong \varprojlim H^p(\hat{A}, R^q\hat{S}(\Omega_e) \otimes L^\vee) = 0.$$

Since the cohomology groups are finite-dimensional linear spaces over k , we may apply Theorem 24 and consider the following spectral sequence

$$\varprojlim H^p(\hat{A}, R^q\hat{S}(\Omega_e) \otimes L^\vee) \Rightarrow \varprojlim H^{p+q}(\hat{A}, R\hat{S}(\Omega_e) \otimes L^\vee).$$

By the discussion above, $\varprojlim H^p(\hat{A}, R^q\hat{S}(\Omega_e) \otimes L^\vee) = 0$ if $p + q > g$. Hence,

$$\varprojlim H^l(\hat{A}, R\hat{S}(\Omega_e) \otimes L^\vee) = 0$$

for any $l > g$.

Since Ω_e and L^\vee are both coherent, we and obtain

$$\begin{aligned} H^l(\hat{A}, R\hat{S}(\Omega_e) \otimes L^\vee) &\cong H^l(A, \Omega_e \otimes RS(L^\vee)) \\ &\cong H^l(A, \Omega_e \otimes RS(D_{\hat{A}}(L)[-g])) \\ &\cong H^{l-g}(A, \Omega_e \otimes RS(D_{\hat{A}}(L))) \\ &\cong H^{l-g}(A, \Omega_e \otimes (-1_A)^* D_A(RS(L))[-g]) \\ &\cong H^{l-g}(A, \Omega_e \otimes (-1_A)^* \hat{L}^\vee), \end{aligned}$$

where the fourth isomorphism is by Lemma 11. Taking the inverse limit, we have

$$0 = \varprojlim H^l(\hat{A}, R\hat{S}(\Omega_e) \otimes L^\vee) \cong \varprojlim H^{l-g}(A, \Omega_e \otimes (-1_A)^* \hat{L}^\vee) \cong H^{l-g}(A, \Omega \otimes (-1_A)^* \hat{L}^\vee),$$

for any $l > g$. The theorem follows. \square

CHAPTER 4

BIRATIONAL GEOMETRY OF IRREGULAR VARIETIES

Remember that in the introduction, we stated two theorems of Ein and Lazarsfeld: a fibration statement for varieties of maximal Albanese dimension and a result characterizing the singularities of theta divisors in principally polarized abelian varieties. We restate them here for convenience.

Theorem 42. (c.f. [10, Theorem 3]; [34, Theorem E]) *If X is a smooth projective variety of maximal Albanese dimension over a field of characteristic zero and $\chi(\omega_X) = 0$, then the image of the Albanese map is ruled by positive-dimensional subtori of A .*

Theorem 43. ([10, Theorem 1] or [30, Theorem 3.5]) *Let A be an abelian variety and let $\Theta \subset A$ be a principal polarization (i.e. an ample divisor such that $h^0(A, \mathcal{O}_A(\Theta)) = 1$). If Θ is irreducible, then it is normal and has rational singularities.*

In this chapter, we illustrate the power of the generic vanishing theory in positive characteristic introduced in Chapter 2 by proving positive characteristic analogues of these two theorems. Concretely, we will be proving the following two results:

Theorem 44. *Let X be a smooth projective variety of maximal Albanese dimension and denote by $a : X \rightarrow A$ the Albanese map. Let $g = \dim A$. Consider the inverse system $\{\Omega_e = F_*^e S^0 a_* \omega_X\}_e$ and denote $\Omega = \varprojlim \Omega_e$. Define $\Lambda_e = RS_{A, \hat{A}} D_A(\Omega_e)$ and assume that the sheaf $\mathcal{H}^0(\Lambda) = \varinjlim \mathcal{H}^0(\Lambda_e)$ has torsion. Then the image of the Albanese map is fibered by abelian subvarieties of \hat{A} .*

The proof of this theorem relies on a technical result that is useful in its own right, so we record it here as well:

Theorem 45. *Let $\{\Omega_e\}$ be an inverse system of coherent sheaves on a g -dimensional abelian variety satisfying the Mittag-Leffler condition and let $\Omega = \varprojlim \Omega_e$. Let $\Lambda_e = RS_{A, \hat{A}}(D_A(\Omega_e))$ and $\Lambda = \varinjlim \Lambda_e$. Suppose that $\{\Omega_e\}$ is a GV-inverse system, in the sense that $\mathcal{H}^i(\Lambda) = 0$*

for any $i \neq 0$. If $\mathcal{H}^0(\Lambda)$ has a torsion point P of dimension $g - k$, then the maps

$$\varprojlim \left(R^k S_{A, \hat{A}}(\Omega_e) \otimes k(P) \right) \rightarrow R^k S_{A, \hat{A}}(\Omega_e) \otimes k(P)$$

are non-zero for every $e \gg 0$.

An identical argument to the one we employ to prove the previous theorem also yields the following result describing the singularities of Theta divisors in positive characteristic.

Theorem 46. *Let A be an ordinary abelian variety over an algebraically closed field of positive characteristic and let Θ be an irreducible Theta divisor. Then Θ is strongly F -regular.*

Over fields of positive characteristic, work of Smith and Hara [41, 17] shows that F -rationality is the positive characteristic counterpart to rational singularities, and the former is implied by strong F -regularity, so in this sense, Theorem 46 is stronger than one might expect.

4.1 Main technical result

Let $\{\Omega_e\}$ be an inverse system of coherent sheaves on a g -dimensional abelian variety satisfying the Mittag-Leffler condition and let $\Omega = \varprojlim \Omega_e$. Let $\Lambda_e = R S_{A, \hat{A}}(D_A(\Omega_e))$ and $\Lambda = \varinjlim \Lambda_e$. Suppose that $\{\Omega_e\}$ is a GV-inverse system, in the sense that $\mathcal{H}^i(\Lambda) = 0$ for any $i \neq 0$. We proved in Theorem 39 that if $\mathcal{H}^0(\Lambda)$ is torsion-free, then the maps

$$\left(\varprojlim R^k S_{A, \hat{A}}(\Omega_e) \right)_P \rightarrow R^k S_{A, \hat{A}}(\Omega_e)_P$$

are zero for any point $P \in \hat{A}$ such that $\dim(P) \geq g - k$. We next show a partial converse to this statement.

In the sequel, we will say that $\mathcal{H}^0(\Lambda)$ has torsion if it is not torsion-free. More concretely, we will say that $\mathcal{H}^0(\Lambda)$ has torsion at a point P if there exists a section $s \in \mathcal{O}_{\hat{A}}$ such that the multiplication map $\mathcal{H}^0(\Lambda)_P \xrightarrow{\times s_P} \mathcal{H}^0(\Lambda)_P$ is not injective.

Before stating our main result we need to introduce some notation. Consider the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Lambda^t & \longrightarrow & \Lambda & \longrightarrow & \mathcal{F} = \Lambda/\Lambda^t \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \tilde{\Lambda}_e^t & \longrightarrow & \tilde{\Lambda}_e & \longrightarrow & \mathcal{F}_e \longrightarrow 0 \end{array} \quad (4.1)$$

where Λ^t denotes the torsion subsheaf of Λ , $\tilde{\Lambda}_e = \text{Im}(\Lambda_e \rightarrow \Lambda)$, $\mathcal{F}_e = \text{Im}(\tilde{\Lambda}_e \rightarrow \mathcal{F})$ and $\tilde{\Lambda}_e^t = \ker(\tilde{\Lambda}_e \rightarrow \mathcal{F})$. It is easy to see that the second row is exact and that $\tilde{\Lambda}_e^t$ is the torsion subsheaf of $\tilde{\Lambda}_e$. It is also clear by construction that $\Lambda = \varinjlim_e \tilde{\Lambda}_e$ (c.f. Theorem 25).

Since the direct limit is exact, by its universal property, there is also a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda^t & \longrightarrow & \Lambda & \longrightarrow & \mathcal{F} = \Lambda/\Lambda^t & \longrightarrow & 0 \\ & & \uparrow & & \uparrow \simeq & & \uparrow & & \\ 0 & \longrightarrow & \varinjlim_e \tilde{\Lambda}_e^t & \longrightarrow & \varinjlim_e \tilde{\Lambda}_e & \longrightarrow & \varinjlim_e \mathcal{F}_e & \longrightarrow & 0 \end{array}$$

Note that $\Lambda^t \simeq \varinjlim_e \tilde{\Lambda}_e^t$. Indeed, an element $\eta \in \Lambda^t \hookrightarrow \Lambda$ lifts to a class $[\tilde{\eta}_e] \in \varinjlim_e \tilde{\Lambda}_e = \Lambda$ and this class maps to 0 under the composition $\varinjlim_e \tilde{\Lambda}_e \rightarrow \mathcal{F}$, so it lies in $\varinjlim_e \tilde{\Lambda}_e^t$ (by exactness of the direct limit). By the 5-lemma, we have that the right vertical map is also an isomorphism.

We are now ready to state our main technical result:

Theorem 47. *Let $\{\Omega_e\}$ be an inverse system of coherent sheaves on a g -dimensional abelian variety satisfying the Mittag-Leffler condition. Let $\Lambda_e = RS_{A,\hat{A}}(D_A(\Omega_e))$ and $\Lambda = \text{hocolim}_e \Lambda_e$. Denote as above $\tilde{\Lambda}_e = \text{Im}(\Lambda_e \rightarrow \Lambda)$ and define $\tilde{\Omega}_e = RS_{\hat{A},A} D_{\hat{A}} \tilde{\Lambda}_e$. Suppose that $\{\Omega_e\}$ is a GV-inverse system, in the sense that $\mathcal{H}^i(\Lambda) = 0$ for any $i \neq 0$. If $\mathcal{H}^0(\Lambda)$ has a torsion point P of maximal dimension $g - k$, then*

$$\varprojlim \left(R^k S_{A,\hat{A}}(\tilde{\Omega}_e) \otimes k(P) \right) \neq 0.$$

Equivalently, the maps

$$\varprojlim \left(R^k S_{A,\hat{A}}(\tilde{\Omega}_e) \otimes k(P) \right) \rightarrow R^k S_{A,\hat{A}}(\tilde{\Omega}_e) \otimes k(P)$$

are non-zero for every $e \gg 0$.

Proof. We start by performing a sequence of reductions.

Reduction 1: With the notation introduced in diagram (4.1), in order to show that

$$\varprojlim \left(R^k S_{A,\hat{A}}(\tilde{\Omega}_e) \otimes k(P) \right) \neq 0$$

it is sufficient to show that

$$\varprojlim_e \left[\mathcal{E}xt^k(\tilde{\Lambda}_e^t, \mathcal{O}_{\hat{A}}) \otimes k(P) \right] \neq 0$$

Indeed, consider the long exact sequence for $\mathcal{E}xt$ induced by the short exact sequence

$$0 \rightarrow \tilde{\Lambda}_e^t \rightarrow \tilde{\Lambda}_e \rightarrow \mathcal{F}_e \rightarrow 0$$

namely

$$\cdots \longrightarrow \mathcal{E}xt^k(\tilde{\Lambda}_e, \mathcal{O}_{\hat{A}}) \longrightarrow \mathcal{E}xt^k(\tilde{\Lambda}_e^t, \mathcal{O}_{\hat{A}}) \longrightarrow \mathcal{E}xt^{k+1}(\mathcal{F}_e, \mathcal{O}_{\hat{A}}) \longrightarrow \cdots$$

By Lemma 6.3 in [35], it follows that $\mathcal{E}xt^{k+1}(\mathcal{F}_e, \mathcal{O}_{\hat{A}}) \otimes k(P) = 0$ for all e , so since $\otimes k(P)$ is right-exact, we obtain a surjection

$$\mathcal{E}xt^k(\tilde{\Lambda}_e, \mathcal{O}_{\hat{A}}) \otimes k(P) \longrightarrow \mathcal{E}xt^k(\tilde{\Lambda}_e^t, \mathcal{O}_{\hat{A}}) \otimes k(P)$$

and hence¹ a surjection

$$\varprojlim_e \left[\mathcal{E}xt^k(\tilde{\Lambda}_e, \mathcal{O}_{\hat{A}}) \otimes k(P) \right] \longrightarrow \varprojlim_e \left[\mathcal{E}xt^k(\tilde{\Lambda}_e^t, \mathcal{O}_{\hat{A}}) \otimes k(P) \right].$$

Therefore, if $\varprojlim_e \left[\mathcal{E}xt^k(\tilde{\Lambda}_e^t, \mathcal{O}_{\hat{A}}) \otimes k(P) \right] \neq 0$, then

$$\varprojlim_e \left[\mathcal{E}xt^k(\tilde{\Lambda}_e, \mathcal{O}_{\hat{A}}) \otimes k(P) \right] \stackrel{[*]}{\simeq} \varprojlim_e \left(R^k S_{A, \hat{A}}(\tilde{\Omega}_e) \otimes k(P) \right) \neq 0$$

as claimed, where the isomorphism $[*]$ follows from the following computation

$$\begin{aligned} \mathcal{E}xt^p(\tilde{\Lambda}_e, \mathcal{O}_{\hat{A}}) &\simeq \mathcal{H}^{p-g} \left(D_{\hat{A}}(\tilde{\Lambda}_e) \right) \simeq \mathcal{H}^{p-g} \left((-1_{\hat{A}})^* R S_{A, \hat{A}} \overbrace{R S_{\hat{A}, A}(D_{\hat{A}}(\tilde{\Lambda}_e))}^{:= \tilde{\Omega}_e} [g] \right) \\ &\simeq \mathcal{H}^p \left((-1_{\hat{A}})^* R S_{A, \hat{A}}(\tilde{\Omega}_e) \right). \end{aligned} \quad (4.2)$$

Reduction 2: Denoting $i : Z := \overline{\{P\}} \hookrightarrow \hat{A}$, we next reduce to showing that

$$\varprojlim_e \left[\mathcal{E}xt^k(Li^* \Lambda_e^t, \mathcal{O}_Z) \otimes k(P) \right] \neq 0$$

In order to see this, it suffices to show that for every e , there is an isomorphism

$$\mathcal{E}xt^k(\tilde{\Lambda}_e^t, \mathcal{O}_{\hat{A}}) \otimes k(P) \simeq \mathcal{E}xt^k(Li^* \tilde{\Lambda}_e^t, \mathcal{O}_Z) \otimes k(P) \quad (4.3)$$

But observe that

$$\begin{aligned} \mathcal{E}xt^k(\tilde{\Lambda}_e^t, \mathcal{O}_{\hat{A}})|_Z \otimes k(P) &\simeq L^0 i^* \mathcal{E}xt^k(\tilde{\Lambda}_e^t, \mathcal{O}_{\hat{A}}) \otimes k(P) \stackrel{[1]}{\simeq} \mathcal{H}^k \left(Li^* R \mathcal{H}om(\tilde{\Lambda}_e^t, \mathcal{O}_{\hat{A}}) \right) \otimes k(P) \\ &\stackrel{[2]}{\simeq} \mathcal{E}xt^k(Li^* \tilde{\Lambda}_e^t, \mathcal{O}_Z) \otimes k(P) \end{aligned}$$

where:

¹Note that the system $\left\{ \mathcal{E}xt^k(\tilde{\Lambda}_e^t, \mathcal{O}_{\hat{A}}) \otimes k(P) \right\}_e$ satisfies the ML-condition.

- (i) The isomorphism in [1] follows from Grothendieck's spectral sequence²

$$E_2^{p,q} = L^p i^* \mathcal{E}xt^q(\tilde{\Lambda}_e^t, \mathcal{O}_{\hat{A}}) \otimes k(P) \implies L^{p+q} i^* R\mathcal{H}om(\tilde{\Lambda}_e^t, \mathcal{O}_{\hat{A}}) \otimes k(P).$$

Note in the first place that $\mathcal{E}xt^q(\tilde{\Lambda}_e^t, \mathcal{O}_{\hat{A}}) = 0$ near P for all $q < k$. Indeed, since $\tilde{\Lambda}_e^t$ is supported on Z^3 near P , which has codimension k , our claim follows from the fact that $\mathcal{E}xt^q(\bullet, \mathcal{O}_{\hat{A}}) = 0$ for all $q < \text{codim Supp } (\bullet)$ (c.f. [35, Lemma 6.1]).

The differentials coming out of $E_2^{0,k} = L^0 i^* \mathcal{E}xt^k(\tilde{\Lambda}_e^t, \mathcal{O}_{\hat{A}})$ are hence trivial and the differential targeting $E_2^{0,k}$ is

$$d_2^{-2,k+1} : L^{-2} i^* \mathcal{E}xt^{k+1}(\tilde{\Lambda}_e^t, \mathcal{O}_{\hat{A}}) \otimes k(P) \longrightarrow L^0 i^* \mathcal{E}xt^k(\tilde{\Lambda}_e^t, \mathcal{O}_{\hat{A}}) \otimes k(P)$$

which is also trivial since there is an open neighborhood U of P such that

$$\left[L^{-2} i^* \mathcal{E}xt^{k+1}(\tilde{\Lambda}_e^t, \mathcal{O}_{\hat{A}}) \right]_U \simeq L^{-2} i_U^* \mathcal{E}xt^{k+1}(\tilde{\Lambda}_e^t, \mathcal{O}_{\hat{A}})_U = 0$$

where $i_U : Z \cap U \hookrightarrow U$ and where the last vanishing follows from the coherence of $\tilde{\Lambda}_e^t$ and the fact that P has codimension k .

- (ii) The isomorphism in [2] follows from the $Lf^* R\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq R\mathcal{H}om(Lf^* \mathcal{F}^\bullet, Lf^* \mathcal{G}^\bullet)$ (c.f. equation (3.17) in [23]).

Finally, in order to show that

$$\varprojlim_e \left(\mathcal{E}xt^k(Li^* \tilde{\Lambda}_e^t, \mathcal{O}_Z) \otimes k(P) \right)$$

is nonzero, we will use the isomorphism

$$\varprojlim_e \left(\mathcal{E}xt_{\mathcal{O}_Z}^k(Li^* \tilde{\Lambda}_e^t, \mathcal{O}_Z) \otimes k(P) \right) = \varprojlim_e \mathcal{E}xt_{k(P)}^k \left(Li^* \tilde{\Lambda}_e^t \otimes k(P), k(P) \right) \neq 0,$$

where we used that $k(P) \simeq \mathcal{O}_P$ and Proposition III.6.8 in [21]. Denote by $i : Z = \overline{\{P\}} \hookrightarrow \hat{A}$ the inclusion. By Grothendieck duality (c.f. discussion in section 2.3 from [4]), since all the higher direct images of a closed immersion are zero, we have a functorial isomorphism

$$R\mathcal{H}om_{\hat{A}}(i_* [Li^* \Lambda_e \otimes k(P)], k(P)[-k]) \simeq Ri_* R\mathcal{H}om_Z \left(Li^* \Lambda_e \otimes k(P), Li^! k(P) \right) \quad (4.4)$$

²C.f. equation (3.10) in [23]. Also note that since tensoring by $k(P)$ is exact on $D(Z)$, there is a spectral sequence as written.

³Note that if Z is an irreducible component of $\text{Supp } \Lambda$, then it is also an irreducible component of $\text{Supp } \mathcal{H}^0(\Lambda_e)$ for every $e \gg 0$. Let Z be one such component, denote by \mathcal{I}_Z its ideal sheaf and take a section $\eta = \{\eta_e\} \in \Lambda$ supported on Z . It is then clear that $Z \subset \text{Supp } \mathcal{H}^0(\Lambda_e)$ for every $e \gg 0$. Now, if $f \in \mathcal{I}_Z$ is in $\text{ann}(\eta)$, it is clear that $f \in \text{ann}(\eta_e)$ for $e \gg 0$, so that $Z = \text{Supp } \eta_e$ for all $e \gg 0$, as claimed.

Taking k -th cohomology, we obtain

$$\begin{aligned} \mathcal{H}om_{k(P)} \left(i_* \left[Li^* \tilde{\Lambda}_e^t \otimes k(P) \right], k(P) \right) &\simeq \mathcal{H}^k \left(Ri_* R\mathcal{H}om_{k(P)} (Li^* \tilde{\Lambda}_e^t \otimes k(P), Li^! k(P)) \right) \\ &\simeq i_* \mathcal{E}xt_{k(P)}^k (Li^* \tilde{\Lambda}_e^t \otimes k(P), Li^! k(P)). \end{aligned} \quad (4.5)$$

Note that the inverse limit of the left-hand side is

$$\varprojlim \mathcal{H}om_{\hat{A}} \left(i_* \left[Li^* \tilde{\Lambda}_e^t \otimes k(P) \right], k(P) \right) = \mathcal{H}om_{\hat{A}} (i_* [Li^* \Lambda \otimes k(P)], k(P)).$$

We claim that the latter sheaf is non-zero. Indeed, note that

$$\mathcal{H}om_{\hat{A}} (i^* \Lambda \otimes k(P), k(P)) \neq 0, \quad (4.6)$$

since it is simply the $k(P)$ -dual of the non-zero $k(P)$ -vector space $\Lambda|_Z \otimes k(P)$. The natural map $Li^* \Lambda \rightarrow L^0 i^* \Lambda$ induces

$$R^0 i_* [Li^* (\Lambda) \otimes k(P)] \xrightarrow{\simeq} R^0 i_* [i^* (\Lambda) \otimes k(P)] \xrightarrow{\neq 0} k(P)$$

where the last map is just a non-zero morphism from (4.6) with the source sheaf extended by zero. Since Λ is locally free in a neighborhood of P and closed immersions have no higher direct images, the fact that the first map is an isomorphism follows from the degeneration of the spectral sequence (c.f. equation (3.10) in [23])

$$R^s i_* (L^t i^* \Lambda \otimes k(P)) \simeq R^s i_* (\mathcal{H}^t (Li^* \Lambda \otimes k(P))) \xrightarrow{s+t=p} R^p i_* (Li^* \Lambda \otimes k(P))$$

since $L^t i^* \Lambda \otimes k(P) = 0$ for all $t < 0$. Hence, the inverse limit on the right-hand side of (4.5) is also non-zero, and in particular that

$$\varprojlim \mathcal{E}xt_Z^k (Li^* \Lambda_e \otimes k(P), Li^! k(P)) \neq 0.$$

But recall that Z is a torsion translate of an abelian subvariety of \hat{A} , so $\omega_Z \simeq \mathcal{O}_Z \simeq \mathcal{O}_P \simeq k(P)$, so $Li^! k(P) \simeq k(P)$ and we may hence conclude that

$$\varprojlim \mathcal{E}xt_Z^k (Li^* \Lambda_e \otimes k(P), k(P)) \neq 0$$

as claimed. □

Remark 48. *Theorem 47 has shown that*

$$\varprojlim \left(R^k S_{A, \hat{A}} (\tilde{\Omega}_e) \otimes k(P) \right) \neq 0.$$

where recall, we defined $\tilde{\Omega}_e = RS_{\hat{A}, A} D_{\hat{A}} \tilde{\Lambda}_e$ where $\tilde{\Lambda}_e = \text{Im}(\Lambda_e \rightarrow \Lambda)$. In what follows, we will drop the tildes in order to ease the notation: all we need is a projective system of coherent sheaves satisfying the generic vanishing property and inducing a non-zero limit as stated in the theorem.

If A has no super-singular factors, we know by Proposition 3.3.5 in [15] that for every $e \geq 0$, the top dimensional components of the set of points $P \in \hat{A}$ such that the map $\mathcal{H}^0(\Lambda_e)_P \rightarrow \mathcal{H}^0(\Lambda)_P$ is non-zero is a torsion translate of an abelian subvariety of \hat{A} .

Let $P \in \hat{A}$ be a torsion point of maximal dimension (namely, $\dim(P)$ is maximal such that $\mathcal{H}^0(\Lambda)_P$ has torsion) and consider $W = \overline{\{P\}}$. In particular, W is a component of $\text{Supp } \mathcal{H}^0(\Lambda)$, and we already argued earlier that W must also be an irreducible component of $\text{Supp } \mathcal{H}^0(\Lambda_e)$ for $e \gg 0$, so W is also a top dimensional component of the support of the image of the map $\mathcal{H}^0(\Lambda_e) \rightarrow \mathcal{H}^0(\Lambda)$. We thus conclude that if $P \in \hat{A}$ is a torsion point of maximal dimension, then $W = \overline{\{P\}}$ is a torsion translate of an abelian subvariety of \hat{A} .

In this context, Theorem 47 yields the following.

Corollary 49. *Let $\{\Omega_e\}$ be a Mittag-Leffler inverse system of coherent sheaves on a g -dimensional abelian variety A with no supersingular factors, and let $\Omega = \varprojlim \Omega_e$. Let $\Lambda_e = RS_{A,\hat{A}}(D_A(\Omega_e))$ and $\Lambda = \text{hocolim} \Lambda_e$. Suppose that $\{\Omega_e\}$ is a GV-inverse system, in the sense that $\mathcal{H}^i(\Lambda) = 0$ for any $i \neq 0$. If P is a torsion point of $\mathcal{H}^0(\Lambda)$ of maximal dimension (so that $W = \overline{\{P\}}$ is a torsion translate of an abelian subvariety of \hat{A}), then there are non-zero maps*

$$\varprojlim \left(R^{g-k} S_{A,W}(\Omega_e) \otimes k(P) \right) \rightarrow R^{g-k} S_{A,W}(\Omega_e) \otimes k(P)$$

for $e \gg 0$, where the Fourier-Mukai kernel of $S_{A,W}$ is given by $\mathcal{P}^{A \times \hat{W}} = (id \times \iota)^* \mathcal{P}^{A \times \hat{A}}$, $\mathcal{P}^{A \times \hat{A}}$ being the normalized Poincaré bundle of $A \times \hat{A}$.

Proof. By Theorem 47, we have a non-zero map

$$\varprojlim \left(R^{g-k} S_{A,\hat{A}}(\Omega_e) \otimes k(P) \right) \rightarrow R^{g-k} S_{A,\hat{A}}(\Omega_e) \otimes k(P)$$

for some $e > 0$. Consider the base change maps

$$R^{g-k} S_{A,\hat{A}}(\Omega_e) \otimes k(P) \rightarrow H^{g-k}(A, \Omega_e \otimes \mathcal{P}_{|A \times \{P\}}^{A \times \hat{A}})$$

and

$$R^{g-k} S_{A,W}(\Omega_e) \otimes k(P) \rightarrow H^{g-k}(A, \Omega_e \otimes \mathcal{P}_{|A \times \{P\}}^{A \times \hat{W}})$$

Note that the second map is an isomorphism by flat base change: indeed, denoting by $\iota : \hat{W} \hookrightarrow \hat{A}$ the inclusion, we have

$$\begin{aligned}
R^{g-k}S_{A,\hat{W}}(\Omega_e) \otimes k(P) &\stackrel{def}{=} R^{g-k}p_{\hat{W}*} \left(p_A^* \Omega_e \otimes \mathcal{P}^{A \times \hat{W}} \right) \otimes k(P) \\
&\stackrel{FBC}{\simeq} R^{g-k}p'_{\hat{W}*} \left(p_A^* \Omega_e \otimes \mathcal{P}^{A \times \hat{W}} \otimes k(P) \right) \\
&\stackrel{def}{\simeq} R^{g-k}p'_{\hat{W}*} \left(p_A^* \Omega_e \otimes (id \times \iota)^* \mathcal{P}^{A \times \hat{A}} \otimes k(P) \right) \\
&\stackrel{[*]}{\simeq} H^{g-k}(A, \Omega_e \otimes \mathcal{P}_{|A \times \{P\}}^{A \times \hat{W}})
\end{aligned}$$

where in $[*]$, we used Proposition III.8.5 in [21] and where $p'_{\hat{W}}$ is the base change of the projection, as illustrated in the diagram

$$\begin{array}{ccc}
A \times_{\hat{W}} \text{Spec } k(P) & \longrightarrow & A \times \hat{W} \\
p'_{\hat{W}} \downarrow & & \downarrow p_{\hat{W}} \\
\text{Spec } k(P) & \xrightarrow{\text{flat}} & \hat{W}
\end{array}$$

However, note that we have

$$H^{g-k}(A, \Omega_e \otimes \mathcal{P}_{|A \times \{P\}}^{A \times \hat{A}}) \simeq H^{g-k}(A, \Omega_e \otimes \mathcal{P}_{|A \times \{P\}}^{A \times \hat{W}}) \quad (4.7)$$

so we can write both base change maps in the following diagram

$$\begin{array}{ccc}
\varprojlim \left(R^{g-k}S_{A,\hat{A}}(\Omega_e) \otimes k(P) \right) & \xrightarrow{\neq 0} & R^{g-k}S_{A,\hat{A}}(\Omega_e) \otimes k(P) \\
\downarrow & & \downarrow [*] \\
\varprojlim H^{g-k} \left(A, \Omega_e \otimes \mathcal{P}_{|A \times \{P\}}^{A \times \hat{A}} \right) & \longrightarrow & H^{g-k} \left(A, \Omega_e \otimes \mathcal{P}_{|A \times \{P\}}^{A \times \hat{A}} \right) \\
\downarrow \simeq & & \downarrow \simeq \\
\varprojlim H^{g-k} \left(A, \Omega_e \otimes \mathcal{P}_{|A \times \{P\}}^{A \times \hat{W}} \right) & \longrightarrow & H^{g-k} \left(A, \Omega_e \otimes \mathcal{P}_{|A \times \{P\}}^{A \times \hat{W}} \right) \\
\uparrow \simeq & & \uparrow \simeq \\
\varprojlim \left(R^{g-k}S_{A,\hat{W}}(\Omega_e) \otimes k(P) \right) & \longrightarrow & R^{g-k}S_{A,\hat{W}}(\Omega_e) \otimes k(P)
\end{array}$$

where the top horizontal map is non-zero for $e \gg 0$ by Theorem 47, the middle isomorphisms are the ones in (4.7), and where the isomorphisms at the bottom follow from flat base change as described above.

We seek to show that the bottom horizontal map is non-zero for some e . Nevertheless, note that if it this were not the case, then the top horizontal map could not possibly be

non-zero, since in any case the base change maps $[*]$ are injective by the proof of Proposition III.12.5 in [21]⁴. \square

Remark 50. We will be using two different Fourier-Mukai kernels on $A \times \hat{W}$. If $\iota : \hat{W} \hookrightarrow \hat{A}$ denotes the inclusion and $\pi : A \rightarrow W$ is the dual projection, we have a diagram

$$\begin{array}{ccc} A \times \hat{W} & \xrightarrow{id_A \times \iota} & A \times \hat{A} \\ \pi \times id_{\hat{W}} \downarrow & & \\ W \times \hat{W} & & \end{array}$$

If $\mathcal{P}^{A \times \hat{A}}$ and $\mathcal{P}^{W \times \hat{W}}$ denote the normalized Poincaré bundles on $A \times \hat{A}$ and $W \times \hat{W}$, respectively, on $A \times \hat{W}$, we may consider the locally-free sheaves $(id_A \times \iota)^* \mathcal{P}^{A \times \hat{A}}$ and $(\pi \times id_{\hat{W}})^* \mathcal{P}^{W \times \hat{W}}$. In corollary 49, we proved a nonvanishing statement for the derived Fourier-Mukai transform with respect to the former kernel and in what follows, we need an analogous statement for the transform with respect to the latter. Nevertheless, note that we are simply looking at fibers over points $w \in \hat{W} \subset \hat{A}$ (concretely over the generic point of \hat{W}), and over these points, both sheaves are isomorphic. Indeed, $w \in \hat{W} \subset \hat{A}$ determines $\mathcal{P}_{|W \times \{w\}}^{W \times \hat{W}} \in \text{Pic}(W)$ and $\mathcal{P}_{|A \times \{w\}}^{A \times \hat{A}} \in \text{Pic}(A)$, with $\mathcal{P}_{|A \times \{w\}}^{A \times \hat{A}} \simeq \pi^* \mathcal{P}_{|W \times \{w\}}^{W \times \hat{W}}$, and therefore

$$\overbrace{\left[(id_A \times \iota)^* \mathcal{P}^{A \times \hat{A}} \right]_{|A \times \{w\}}}^{\simeq \mathcal{P}_{|A \times \{w\}}^{A \times \hat{A}}} \simeq \overbrace{\left[(\pi \times id_{\hat{W}})^* \mathcal{P}^{W \times \hat{W}} \right]_{|A \times \{w\}}}^{\simeq \pi^* \mathcal{P}_{|W \times \{w\}}^{W \times \hat{W}}}$$

4.2 Fiberings of the Albanese image

Our goal in this section is to prove Theorem 44, which we recall is a positive characteristic analogue of Ein and Lazarsfeld's well-known Theorem 42.

Let X be a smooth projective variety of maximal Albanese dimension and denote by $a : X \rightarrow A$ the Albanese map. Then $a_* \omega_X$ is a Cartier module and we may consider the

⁴In a nutshell, let $f : X \rightarrow Y = \text{Spec } A$ be a projective morphism and let \mathcal{F} be a coherent sheaf on X . For any A -module M , define $T^i(M) := H^i(X, \mathcal{F} \otimes_A M)$, which is a covariant additive functor from A -modules to A -modules which is exact in the middle (by Proposition III.12.1 in [21]). Writing

$$A^r \rightarrow A^s \rightarrow M \rightarrow 0$$

we have a diagram

$$\begin{array}{ccccccc} T^i(A) \otimes A^r & \longrightarrow & T^i(A) \otimes A^s & \longrightarrow & T^i(A) \otimes M & \longrightarrow & 0 \\ \simeq \downarrow & & \simeq \downarrow & & \downarrow \varphi & & \\ T^i(A^r) & \longrightarrow & T^i(A^s) & \longrightarrow & T^i(M) & & \end{array}$$

where $\varphi : T^i(A) \otimes M \rightarrow T^i(M)$ is the base change map and where the two first vertical arrows are isomorphisms. A straight-forward diagram chase then shows that φ is injective.

inverse system $\{\Omega_e = F_*^e S^0 a_* \omega_X\}_e$. Define $\Lambda_e = RS_{A, \hat{A}} D_A(\Omega_e)$ and set $\Lambda = \varinjlim \Lambda_e$. By Corollary 26(ii), we have that $H^i(\Omega \otimes P_\alpha) = 0$ for every $i > 0$ and very general $\alpha \in \hat{A}$. Thus, defining as above $\chi(\Omega) := \chi(\Omega \otimes P_\alpha)$ for very general $\alpha \in \hat{A}$, we see that $\chi(\Omega) = h^0(\Omega \otimes P_\alpha)$ and it seems that in trying to extend Theorem 7 to positive characteristic, one should assume that $h^0(\Omega \otimes P_\alpha) = 0$.

This leaves us in a setting which is similar to the one we encountered in the proof of Theorem 47. If $rk(\Lambda) = h^0(\Omega \otimes P_\alpha) = 0$, then in particular, Λ must be a torsion sheaf. In light of this observation, we show the following:

Theorem 51. *Let X be a smooth projective variety of maximal Albanese dimension and let $a : X \rightarrow A$ be a generically finite map to an abelian variety A with no supersingular factors. Let $g = \dim A$. Consider the inverse system $\{\Omega_e = F_*^e S^0 a_* \omega_X\}_e$ and denote $\Omega = \varprojlim \Omega_e$. Define $\Lambda_e = RS_{A, \hat{A}} D_A(\Omega_e)$ and assume that the sheaf $\mathcal{H}^0(\Lambda) = \varinjlim \mathcal{H}^0(\Lambda_e)$ has torsion. Then the image of the Albanese map is fibered by linear subvarieties of \hat{A} .*

Proof. Let $w \in \hat{A}$ be a torsion point of $\mathcal{H}^0(\Lambda)$ of maximal dimension k ; by the remark preceding Corollary 49, we have that $\hat{W} := \overline{\{w\}} \subset \hat{A}$ is a torsion translate of an abelian subvariety of \hat{A} , which we still denote by \hat{W} . Denote by $\pi : A \rightarrow W$ the projection dual to the inclusion $\hat{W} \hookrightarrow \hat{A}$.

$$\begin{array}{ccc} Y := a(X) & \longrightarrow & A \\ h = \pi|_Y \downarrow & \swarrow \pi & \\ W & & \end{array}$$

By Corollary 49, we know that the map

$$\varprojlim \left(R^{g-k} S_{A, \hat{W}}(\Omega_e) \otimes k(w) \right) \rightarrow R^{g-k} S_{A, \hat{W}}(\Omega_e) \otimes k(w)$$

is non-zero for every $e \gg 0$, where the Fourier-Mukai kernel of $S_{A, \hat{W}}$ is given by $(id \times \iota)^* \mathcal{P}$, \mathcal{P} being the normalized Poincaré bundle of $A \times \hat{A}$.

Recall that, in general, even though the system $\{\Omega_e\}$ satisfies the Mittag-Leffler condition, the inverse system $\{R^t S(\Omega_e)\}_e$ may fail to do so (c.f. Example 3.2 in [42]). We handle the Mittag-Leffler case first, however, since the proof is neater and the subsequent generalization does not rely on new ideas.

Case in which $\{R^{g-k} S_{A, \hat{W}}(\Omega_e)\}_e$ satisfies the ML-condition. The proof in this case goes along the lines of that of Theorem E in [34]. Note that if w is the generic point of $\hat{W} \hookrightarrow \hat{A}$, we have the following:

$$\begin{aligned}
w &\stackrel{[1]}{\in} \left\{ w \in \hat{W} : \varprojlim_e \left(R^{g-k} S_{A, \hat{W}}(\Omega_e) \otimes k(w) \right) \xrightarrow{\neq 0} R^{g-k} S_{A, \hat{W}}(\Omega_e) \otimes k(w) \right\} \\
&\stackrel{[2]}{=} \left\{ w \in \hat{W} : \left(\varprojlim_e R^{g-k} S_{A, \hat{W}}(\Omega_e) \right) \otimes k(w) \xrightarrow{\neq 0} R^{g-k} S_{A, \hat{W}}(\Omega_e) \otimes k(w) \right\} \\
&\subseteq \left\{ w \in \hat{W} : \left(\varprojlim_e R^{g-k} S_{A, \hat{W}}(\Omega_e) \right)_w \xrightarrow{\neq 0} \left(R^{g-k} S_{A, \hat{W}}(\Omega_e) \right)_w \right\} \\
&\subseteq \hat{W}
\end{aligned}$$

where w lies in the first set by Corollary 49 and where the equality [2] follows from Lemma 21, since we are under the assumption that the system $\{R^{g-k} S_{A, \hat{W}}(\Omega_e)\}_e$ satisfies the Mittag-Leffler condition.

This implies that the codimension (in \hat{W}) of the support of image of the map

$$\varprojlim_e R^{g-k} S_{A, \hat{W}}(\Omega_e) \longrightarrow R^{g-k} S_{A, \hat{W}}(\Omega_e)$$

is zero (this support is closed - under the Mittag-Leffler assumption - by Proposition 4.3 in [42]). At the same time, by Proposition 30, we know that this codimension must be $\geq g - k - f$, where f is the dimension of a general fiber of h , so in particular $f \geq g - k$ and this concludes the proof under the Mittag-Leffler assumption (indeed, the fibers of the projection $A \rightarrow W$ are abelian subvarieties of dimension $g - k$).

General case. We finally observe that, in our setting, we can actually do without the Mittag-Leffler assumption on $\{R^{g-k} S_{A, \hat{W}}(\Omega_e)\}_e$. We only used this assumption in order to guarantee the closedness of the support of the image of the map $\left(\varprojlim_e R^{g-k} S_{A, \hat{W}}(\Omega_e) \right)_w \xrightarrow{\neq 0} \left(R^{g-k} S_{A, \hat{W}}(\Omega_e) \right)_w$ and in order to ensure that the inverse limit commutes with $\otimes k(w)$.

Note in the first place that we do not need the support of the image of the above map to be closed for the previous argument to work. Proposition 30 shows that in order for w to belong to the support, we need $\text{codim } \overline{\{w\}} \geq g - k - f$, and this suffices in order to conclude that $f \geq g - k$.

With regards to the commutation of the inverse limit and $\otimes k(w)$, it follows from the flatness of $k(w)$ as an $\mathcal{O}_{\hat{W}}$ -module and the fact that the projection formula and its consequences still hold in the category of quasi-coherent sheaves under some perfection assumptions (c.f. Lemma 71 in [29]). We state this below as a lemma, and the proof is hence complete. \square

Lemma 52. *With the same notations as above, if*

$$\varprojlim_e \left(R^{g-k} S_{A, \hat{W}}(\Omega_e) \otimes k(w) \right) \xrightarrow{\neq 0} R^{g-k} S_{A, \hat{W}}(\Omega_e) \otimes k(w),$$

then

$$\varprojlim_e \left(R^{g-k} S_{A, \hat{W}}(\Omega_e) \right) \otimes k(w) \xrightarrow{\neq 0} R^{g-k} S_{A, \hat{W}}(\Omega_e) \otimes k(w).$$

In particular, equality [2] in the above chain still holds.

Proof. As in the proof of Proposition 30, let $\tilde{\Lambda}_e = R\tilde{S}_{A, \hat{W}}(D_A(\Omega_e))$, where $\tilde{S}_{A, \hat{W}}$ denotes the Fourier-Mukai transform with kernel \mathcal{P}^\vee , with $\mathcal{P} = (\pi \times 1_{\hat{W}})^* \mathcal{P}^{W \times \hat{W}}$. Note in the first place that we have the following isomorphisms of $\mathcal{O}_{\hat{W}}$ -modules:

$$\begin{aligned} \varprojlim_e \left(R^{g-k} S_{A, \hat{W}}(\Omega_e) \otimes k(w) \right) &\stackrel{[1]}{\simeq} \varprojlim_e \left(\mathcal{E}xt_{\mathcal{O}_{\hat{W}}}^{g-k}(\tilde{\Lambda}_e, \mathcal{O}_{\hat{W}}) \otimes k(w) \right) \\ &\simeq \varprojlim_e \left(\mathcal{H}^{g-k} \left(R\mathcal{H}om_{\mathcal{O}_{\hat{W}}}(\tilde{\Lambda}_e, \mathcal{O}_{\hat{W}}) \right) \otimes k(w) \right) \\ &\stackrel{[2]}{\simeq} \varprojlim_e \mathcal{H}^{g-k} \left(R\mathcal{H}om_{\mathcal{O}_{\hat{W}}}(\tilde{\Lambda}_e, \mathcal{O}_{\hat{W}}) \otimes k(w) \right) \\ &\stackrel{[3]}{\simeq} \mathcal{H}^{g-k} \left(\varprojlim_e \left(R\mathcal{H}om_{\mathcal{O}_{\hat{W}}}(\tilde{\Lambda}_e, \mathcal{O}_{\hat{W}}) \otimes k(w) \right) \right) \\ &\stackrel{[4]}{\simeq} \mathcal{H}^{g-k} \left(\varprojlim_e R\mathcal{H}om_{\mathcal{O}_{\hat{W}}}(\tilde{\Lambda}_e, k(w)) \right) \\ &\stackrel{[5]}{\simeq} \mathcal{H}^{g-k} \left(R\mathcal{H}om_{\mathcal{O}_{\hat{W}}}(\tilde{\Lambda}, k(w)) \right) \\ &\stackrel{[6]}{\simeq} \mathcal{H}^{g-k} \left(R\mathcal{H}om_{\mathcal{O}_{\hat{W}}}(\tilde{\Lambda}, \mathcal{O}_{\hat{W}}) \otimes k(w) \right) \\ &\stackrel{[7]}{\simeq} \mathcal{H}^{g-k} \left(R\mathcal{H}om_{\mathcal{O}_{\hat{W}}}(\tilde{\Lambda}, \mathcal{O}_{\hat{W}}) \right) \otimes k(w) \\ &\stackrel{[8]}{\simeq} \mathcal{H}^{g-k} \left(\varprojlim_e RS_{A, \hat{W}}(\Omega_e) \right) \otimes k(w) \end{aligned}$$

where [1] and [8] follow from the computations (3.1) and (3.2) in the proof of Theorem 39, [2] and [7] follow from the flatness of $\otimes k(w)$ as an $\mathcal{O}_{\hat{W}}$ -module, [3] follows from Proposition 20, since the system $\{\mathcal{E}xt^{p-1}(\Lambda_e, \mathcal{O}_{\hat{W}}) \otimes k(w)\}_e$ satisfies the ML-condition, and [4] and [6] follow from the isomorphism⁵

$$R\mathcal{H}om(\mathcal{F}, \mathcal{G}) \otimes \mathcal{H} \simeq R\mathcal{H}om(\mathcal{F}, \mathcal{G} \otimes \mathcal{H}).$$

Hence, by assumption, we have a non-zero map

$$\mathcal{H}^{g-k} \left(\varprojlim_e RS_{A, \hat{W}}(\Omega_e) \right) \otimes k(w) \xrightarrow{\neq 0} R^{g-k} S_{A, \hat{W}}(\Omega_e) \otimes k(w)$$

⁵This isomorphism holds for complexes of sheaves of modules $\mathcal{F}, \mathcal{G}, \mathcal{H}$ provided that either \mathcal{F} or \mathcal{H} are perfect (c.f. Lemma 71 in [29]). Note that $k(w)$ is a perfect complex, being a coherent sheaf.

and the conclusion of the lemma then follows from the following commutative diagram

$$\begin{array}{ccc}
 \varprojlim_e \left(R^{g-k} S_{A, \hat{W}}(\Omega_e) \right) \otimes k(w) & \longrightarrow & R^{g-k} S_{A, \hat{W}}(\Omega_e) \otimes k(w) \\
 \uparrow & \nearrow \neq 0 & \\
 \mathcal{H}^{g-k} \left(\varprojlim_e R S_{A, \hat{W}}(\Omega_e) \right) \otimes k(w) & &
 \end{array}$$

□

In particular, within the context of principally polarized abelian varieties, the same argument yields the following statement:

Corollary 53. *Let (A, Θ) is a principally polarized abelian variety with no supersingular factors defined over an algebraically closed field of characteristic $p > 0$. Assume further that Θ is irreducible. Consider the inverse system $\{F_*^e(\omega_\Theta \otimes \tau_\Theta)\}_e$ on A and set $\Lambda = \varprojlim_e R S_{A, \hat{A}} D_A(\Omega_e)$. Then Λ is a torsion-free quasi-coherent sheaf concentrated in degree 0.*

Proof. The fact that $\Lambda = \mathcal{H}^0(\Lambda)$ is a quasi-coherent sheaf concentrated in degree zero follows from Theorem 25(i), since ω_Θ is a Cartier module.

Assume for a contradiction that $\mathcal{H}^0(\Lambda)$ is not torsion-free and fix an irreducible component $\hat{W} := \overline{\{w\}} \hookrightarrow \hat{A}$ of maximal dimension of the closure of the set of torsion points of $\mathcal{H}^0(\Lambda)$. Denote by $\pi : A \twoheadrightarrow W$ the dual projection.

$$\begin{array}{ccc}
 \Theta & \hookrightarrow & A \\
 h \downarrow & \searrow \pi & \\
 W & &
 \end{array}$$

We may then argue as in the proof of Theorem 51 to conclude that Θ is fibered by abelian subvarieties of A , but this is not possible given that Θ is irreducible (and hence of general type) in light of Abramovich's work (c.f. [1]). □

4.3 Singularities of Theta divisors

We now focus on the singularities of Theta divisors and embark on the proof of Theorem 46. As a warm-up, we focus on simple abelian varieties to start with, namely those which do not contain smaller dimensional abelian varieties.

4.3.1 Case of simple abelian varieties

The crux of the argument resides in the construction of sections of $\mathcal{O}_A(\Theta)$ which vanish along the test ideal $\tau(\Theta)$ and, in the case of simple abelian varieties, it is a direct consequence

of the results in [15]. The proof of the general case will follow the same pattern, albeit further work will be required to prove that the required sections exist.

Theorem 54. *Let (A, Θ) be a PPAV over an algebraically closed field K of characteristic $p > 0$ such that A is simple and ordinary. Then Θ is strongly F -regular. (In particular, Θ is F -rational, and by [4, Lemma 2.34], it is normal and Cohen-Macaulay).*

Proof. On \hat{A} consider the inverse system $\Omega_e = F_*^e \Omega_0$, where $\Omega_0 = \omega_\Theta \otimes \tau(\Theta)$. This yields a direct system $\Lambda_e = R\hat{S}D_A \Omega_e$ equipped with natural maps $\mathcal{H}^0(\Lambda_e) \rightarrow \mathcal{H}^0(\Lambda) = \Lambda = \varinjlim \Lambda_e$. By 25, we know that Λ is quasi-coherent sheaf in degree 0.

Consider the set

$$Z = \left\{ \alpha \in \hat{A} : \operatorname{Im} \left(\mathcal{H}^0(\Lambda_0) \otimes \mathcal{O}_{\hat{A}, \alpha} \rightarrow \Lambda \otimes \mathcal{O}_{\hat{A}, \alpha} \right) \neq 0 \right\}.$$

By Proposition 27, we know that Z is a finite union of torsion translates of subtori. Since \hat{A} is simple by assumption, this implies that either $Z = \hat{A}$ or Z is a finite set.

Assume for a contradiction that Z is finite. By Proposition 28, we have $t_x^* \Omega = \Omega$ for every $x \in \widehat{\hat{A}/Z} = \hat{\hat{A}}$, so that $\operatorname{Supp} \Omega = A$. Since the maps in the inverse system $F_*^e \Omega_0 = F_*^e(\omega_\Theta \otimes \tau(\Theta))$ are surjective, we know by Proposition 28 that $\operatorname{Supp} \Omega = \operatorname{Supp} \Omega_0 = \operatorname{Supp} \omega_\Theta \otimes \tau(\Theta) = A$, which is absurd.

We must thus have $Z = \hat{A}$, so that $\operatorname{Supp} \mathcal{H}^0(\Lambda_0) = \hat{A}$, and hence cohomology and base change yields $H^0(A, \omega_\Theta \otimes \tau(\Theta) \otimes P_\alpha) \neq 0$ for all $\alpha \in \hat{A}$.

Consider the following commutative diagram:

$$\begin{array}{ccccccc} H^0(A, P_\alpha) & \longrightarrow & H^0(A, \mathcal{O}_A(\Theta) \otimes P_\alpha) & \longrightarrow & H^0(\Theta, \mathcal{O}_A(\Theta)|_\Theta \otimes P_\alpha) & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow & & \\ H^0(A, K \otimes P_\alpha) & \longrightarrow & H^0(A, \mathcal{O}_A(\Theta) \otimes P_\alpha \otimes \tilde{\tau}) & \longrightarrow & H^0(\Theta, \mathcal{O}_A(\Theta)|_\Theta \otimes \tau(\Theta) \otimes P_\alpha) & & \end{array}$$

In the top row, we have $H^1(A, P_\alpha) = 0$ for $\alpha \neq 0$ since P_α is topologically trivial. The polarization induced by Θ is principal, so $h^0(\mathcal{O}_A(\Theta) \otimes P_\alpha) = 1$. Since by the above discussion $H^0(\Theta, \mathcal{O}_A(\Theta)|_\Theta \otimes P_\alpha \otimes \tau(\Theta)) \neq 0$, it follows that $H^0(\Theta, \mathcal{O}_A(\Theta)|_\Theta \otimes P_\alpha) \neq 0$ and hence both the right inclusion and the top right restriction are equalities.

The ideal sheaf $\tilde{\tau}$ on A is defined as follows: fix an open subset $U = \operatorname{Spec} R \subseteq A$ and assume that Θ is given by an ideal sheaf $I = I(\Theta)$. Let $J = \tau_\Theta(U)$ be the test ideal of Θ and let $\tilde{J} \subset R$ be an ideal such that $J = \tilde{J}/I$. Omitting the twist by P_α , the diagram above locally boils down to

$$\begin{array}{ccccccc}
& & & R/\tilde{J} \xrightarrow{\simeq} (R/I)/(\tilde{J}/I) \simeq R/\tilde{J} & & & \\
& & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I & \longrightarrow & R & \longrightarrow & R/I \longrightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & I & \longrightarrow & \tilde{J} & \longrightarrow & J \longrightarrow 0
\end{array}$$

so $\tilde{\tau}(U) = \tilde{J}$. Now taking cohomology, a section $s \in H^0(J)$ embeds as $\bar{s} \in H^0(R/I)$ and maps to zero in $H^0(R/\tilde{J})$ by exactness. By exactness of the second row, \bar{s} lifts to $\tilde{\bar{s}} \in H^0(R)$, which still projects to zero in $H^0(R/\tilde{J})$ by commutativity of the top square, so $\tilde{\bar{s}}$ must lift to a non-zero section of $H^0(\tilde{J})$.

Finally, since $H^0(\Theta, \mathcal{O}_A(\Theta)|_{\Theta} \otimes P_{\alpha} \otimes \tau(\Theta)) \neq 0$ for every $\alpha \in \hat{A}$ and these sections lift to sections of $H^0(A, \mathcal{O}_A(\Theta) \otimes P_{\alpha})$ vanishing along $\tilde{\tau}$, we conclude that $h^0(A, \mathcal{O}_A(\Theta) \otimes P_{\alpha} \otimes \tilde{\tau}) = 1$. Hence, if $\tilde{\tau}$ were not trivial, we would have $\text{Zeros}(\tilde{\tau}) \subset \Theta + \alpha_P$ for every $\alpha_P \in A$ (where $\alpha_P \in A$ is the point corresponding to $P_{\alpha} \in \text{Pic}^0(A)$), which is absurd since these translates of Θ do not have any points in common. We thus conclude that $\tilde{\tau} = \mathcal{O}_A$, and hence $\tau(\Theta) = \mathcal{O}_{\Theta}$ so that Θ is strongly F-regular. \square

In the proof of Theorem 54, we used the simplicity of A in order to show that $H^0(A, \omega_{\Theta} \otimes \tau(\Theta) \otimes P_{\alpha}) \neq 0$ for all $\alpha \in \hat{A}$. The same argument we employed above will work in the general case provided that we can show the existence of sections in $H^0(A, \omega_{\Theta} \otimes \tau(\Theta) \otimes P_{\alpha})$, and it turns out that this is somewhat more involved.

4.3.2 General case

We finally study singularities of Theta divisors in the general setting. As we mentioned earlier, the argument will be analogous to the one employed to prove the theorem in the case of simple abelian varieties, although additional work will be required to prove that there exist sections in $H^0(\Theta, \mathcal{O}_A(\Theta)|_{\Theta} \otimes \tau(\Theta) \otimes P_{\alpha})$.

The main ingredient in Ein and Lazarsfeld's proof over fields of characteristic zero was that given a smooth projective variety X of maximal Albanese dimension such that $\chi(X, \omega_X) = 0$, the image of its Albanese morphism is fibered by tori (c.f. Theorem 3 in [10]). Our proof will rely on Corollary 53, where we proved that if the sheaf $\mathcal{H}^0(\Lambda)$ associated to the inverse system $\{F_*^e(\omega_{\Theta} \otimes \tau_{\Theta})\}$ was not torsion-free, then Θ would be fibered by tori, which is impossible since Θ is irreducible.

In a nutshell, and as in the case of simple abelian varieties, Θ will be strongly F-regular provided that there exist nontrivial sections in $H^0(\Theta, \mathcal{O}_A(\Theta) \otimes \mathcal{O}_\Theta \otimes \tau(\Theta) \otimes P_\alpha)$ and we will show that if that was not the case, then the sheaf $\mathcal{H}^0(\Lambda)$ would have torsion, a contradiction.

Theorem 55. *Let (A, Θ) be an ordinary principally polarized abelian variety over an algebraically closed field k of characteristic $p > 0$. If Θ is irreducible, then Θ is strongly F-regular.*

Proof. The proof goes along the lines of Theorem 54: consider again the commutative diagram:

$$\begin{array}{ccccccc} H^0(A, P_\alpha) & \longrightarrow & H^0(A, \mathcal{O}_A(\Theta) \otimes P_\alpha) & \longrightarrow & H^0(\Theta, \mathcal{O}_A(\Theta)|_\Theta \otimes P_\alpha) & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow & & \\ H^0(A, K \otimes P_\alpha) & \longrightarrow & H^0(A, \mathcal{O}_A(\Theta) \otimes P_\alpha \otimes \tilde{\tau}) & \longrightarrow & H^0(\Theta, \mathcal{O}_A(\Theta)|_\Theta \otimes \tau(\Theta) \otimes P_\alpha) & & \end{array}$$

In the proof of Theorem 54, we used the simplicity of A to conclude easily that

$$H^0(\Theta, \overbrace{\mathcal{O}_A(\Theta) \otimes \mathcal{O}_\Theta}^{\omega_\Theta} \otimes \tau(\Theta) \otimes P_\alpha) \neq 0$$

It then followed from the commutative diagram above that $H^0(A, \mathcal{O}_A(\Theta) \otimes P_\alpha \otimes \tilde{\tau}) \neq 0$ and this in turn forced $\tilde{\tau}$ to be trivial, whence $\tau(\Theta) = \mathcal{O}_\Theta$.

We shall now use the previous results in order to conclude that $H^0(\Theta, \omega_\Theta \otimes \tau(\Theta) \otimes P_\alpha) \neq 0$. In a nutshell, assuming for a contradiction that $\tau(\Theta)$ is not trivial, we will show that $0 \neq S^0(\Theta, \omega_\Theta \otimes \tau(\Theta) \otimes P_\alpha) \subseteq H^0(\Theta, \omega_\Theta \otimes \tau(\Theta) \otimes P_\alpha)$ for general $\alpha \in \hat{A}$. The diagram above then yields $H^0(A, \mathcal{O}_A(\Theta) \otimes P_\alpha \otimes \tilde{\tau}) \neq 0$ and since by assumption $\tau(\Theta) \neq \mathcal{O}_\Theta$, we conclude that $\text{Zeroes}(\tilde{\tau}) \subset \Theta + \alpha_P$ for general $\alpha_P \in A$ (as before, α_P denotes the point in A corresponding to $P_\alpha \in \text{Pic}^0 A$), but this is not possible since general translates of Θ do not have points in common. Therefore, we must have $\tau(\Theta) = \mathcal{O}_\Theta$, and hence Θ is strongly F-regular.

We now argue by contradiction: let $\Omega = \varprojlim F_*^e S^0 \omega_\Theta = \varprojlim F_*^e (\omega_\Theta \otimes \tau(\Theta))$. By Corollary 3.2.1 in [15] and its proof, for all closed $\alpha \in \hat{A}$ we have that

$$\mathcal{H}^0(\Lambda) \otimes k(\alpha) \simeq H^0(\Theta, \varprojlim \Omega_e \otimes P_\alpha^\vee)^\vee$$

so assuming for a contradiction that $S^0(\Theta, \omega_\Theta \otimes \tau(\Theta) \otimes P_\alpha) = 0$ for general $\alpha \in \hat{A}$, it follows that $rk(\Lambda) = 0$, and hence that $\mathcal{H}^0(\Lambda)$ has torsion. Nevertheless, this is not possible by Corollary 53, since we are assuming that Θ is irreducible, and this concludes the proof. \square

CHAPTER 5

PLURICANONICAL MAPS

Let X be a nonsingular projective variety of general type over an algebraically closed field k , $n = \dim(X)$ and K_X be a canonical divisor. Since K_X is big, for any sufficiently large positive integer m , the linear series $|mK_X|$ induces a birational map and bounding this integer m is a problem that has received considerable attention. One of the most recent results in this direction (in characteristic zero) is the following theorem of Jiang, Lahoz and Tirabassi.

Theorem 56. *[24, Theorem A] Let X be a smooth projective variety of general type and maximal Albanese dimension, over a field of characteristic zero. Then the linear system $|3K_X|$ induces a birational map.*

In positive characteristic, it was only recently that Zhang showed, under a separability assumption on the Albanese map, that the fourth canonical divisors induces a birational map. Concretely, by employing techniques of Chen and Hacon [9] consistent of repeatedly using the Fourier-Mukai transform to produce sections of $|mK_X|$, Zhang proved the following:

Theorem 57. *[43, Theorem 5.6] Let X be a smooth projective variety of general type over an algebraically closed field of positive characteristic. If X has maximal Albanese dimension and the Albanese map is separable, then the linear system $|4K_X|$ induces a birational map.*

Our goal in this chapter, which contains preliminary work only, is to try to couple the generic vanishing techniques outlined so far with the methodology in [9, 24, 43] to extend Theorem 56 to positive characteristic. Concretely, our goal is to shed some light on the proof of the following result:

Theorem 58. *Let X be a smooth projective variety of general type over an algebraically closed field of positive characteristic. If X has maximal Albanese dimension and the Albanese map is separable, then the linear system $|3K_X|$ induces a birational map.*

The chapter is structured as follows: in section 5.1, we outline the proof of the theorem in characteristic zero (following [24] for the case $\chi(\omega_X) = 0$ and [34] for the case $\chi(\omega_X) > 0$). In section 5.2, we introduce the notation from [43] and we outline the vanishing results proved therein. Finally, in section 5.3.1, we give a proof of Theorem 58 when the image of the Albanese map is ruled by Abelian subvarieties (which corresponds to the characteristic zero case $\chi(\omega_X) > 0$, [9, 34]) and in section 5.3.2, we outline how one could implement the approach in [24] in positive characteristic, we prove some results in this direction and we remark the main difficulty that we have encountered.

5.1 Outline of the proof in characteristic zero

In this section, we outline the proof of Theorem 56. As remarked in the preceding introduction, the case $\chi(\omega_X) > 0$ is due to Chen and Hacon [9], whereas the case $\chi(\omega_X) = 0$ was proven in [24], using different techniques. In order to extend the first case, it is easier for us work with Pareschi and Popa's approach to the problem in [34], so we start by outlining their proof. Since they used the notion of continuous global generation, we begin with a brief introduction following [32].

5.1.1 Case $\chi(\omega_X) > 0$

Recall that a coherent sheaf \mathcal{F} on an abelian variety A is a GV sheaf if

$$\text{codim Supp } R^i \hat{S}(\mathcal{F}) \geq i$$

for every $i > 0$ and it is M -regular if $\text{codim Supp } R^i \hat{S}(\mathcal{F}) > i$ for every $i > 0$. The next lemma collects some characterizations of M -regular sheaves:

Lemma 59. *Let \mathcal{F} be a coherent sheaf on an abelian variety A . The following conditions are equivalent:*

- (i) *$\text{codim Supp } R^i \hat{S}(\mathcal{F}) > i$ for all $i > 0$.*
- (ii) *$\text{codim } V^i(\mathcal{F}) > i$ for all $i > 0$.*
- (iii) *$R\hat{S}(D_A(\mathcal{F}))$ is a torsion free sheaf (it is a sheaf, since it is GV).*
- (iv) *For every locally free sheaf H on X satisfying IT_0^1 , and for every non-empty Zariski open set $U \subset \hat{A}$, the sum of multiplication maps of global sections*

¹Namely, $H^i(A, H \otimes P_\alpha) = 0$ for all $i > 0$ and all $\alpha \in \hat{A}$.

$$\mathcal{M}_U : \bigoplus_{\alpha \in \hat{A}} H^0(A, \mathcal{F} \otimes P_\alpha) \otimes H^0(A, H \otimes P_\alpha^\vee) \xrightarrow{\oplus m_\alpha} H^0(A, \mathcal{F} \otimes H)$$

is surjective.

Definition 60 (Continuous global generation). *A sheaf \mathcal{F} on X is **continuously globally generated** with respect to $a : X \rightarrow A$ if the sum of evaluation maps*

$$Ev_U = \bigoplus_{\alpha \in U} H^0(X, \mathcal{F} \otimes a^* P_\alpha) \otimes P_\alpha^\vee \longrightarrow \mathcal{F}$$

is surjective for every nonempty open subset $U \subset \hat{A}$. If $T \subset X$ is a proper subvariety, we say that \mathcal{F} is **continuously globally generated away from T** if $\text{Supp}(\text{coker } Ev_U) \subset T$ for every $U \subset \hat{A}$.

The following are the two main properties of CGG sheaves:

Proposition 61. [32, Propositions 2.12, Proposition 2.13] *Let \mathcal{F} be a coherent sheaf on an abelian variety.*

- (i) [**Global generation**]. *If \mathcal{F} and some line bundle L on X are both CGG, then $\mathcal{F} \otimes L$ is globally generated.*
- (ii) [**M-regular sheaves are CGG**]. *If \mathcal{F} is M-regular, there exists $N > 0$ such that for general $\alpha_1, \dots, \alpha_N \in \hat{A}$, the sum of evaluation maps*

$$\bigoplus_{i=1}^N H^0(A, \mathcal{F} \otimes P_{\alpha_i}) \otimes P_{\alpha_i}^\vee \longrightarrow \mathcal{F}$$

is surjective.

As an example of these notions, we have the following theorem:

Theorem 62. [34, Proposition 5.5] *Let X be a smooth projective variety of maximal Albanese dimension, $a : X \rightarrow A$ be the Albanese map and denote by Z the exceptional locus of a . Assume that $a(X)$ is not ruled by abelian subvarieties of A . Then:*

- (i) $a_* \omega_X$ is M-regular.
- (ii) $a_* \omega_X$ is continuously globally generated.
- (iii) ω_X is continuously globally generated away from Z .
- (iv) For all $k \geq 2$, $\omega_X^k \otimes a^* P_\alpha$ is globally generated away from Z for any $\alpha \in \text{Pic}^0(X)$.

Proof. Recall that if $a : X \rightarrow A$ is generically finite, then $\text{codim } V_a^i(\omega_X) \geq i$ for every $i \geq 0$ and the sets $V_a^1(\omega_X)$ are unions of torsion translates of subtori of \hat{A} . They also showed that if there exists an i such that $\text{codim } V_a^i(\omega_X) = i$, then the image $a(X)$ is ruled by subtori of A .

- (i) By Grauert-Riemenschneider vanishing, we know $R^i a_* \omega_X = 0$ for every $i > 0$ so that $V_a^i(\omega_X) = V^i(a_* \omega_X)$ for every $i > 0$. Since $a(X)$ is not ruled by subtori of A , Ein-Lazarsfeld's result implies that ω_X is M-regular, and hence so is $a_* \omega_X$.
- (ii) M-regularity implies continuous global generation by Proposition 61(ii) above.
- (iii) Continuous global generation is preserved by finite maps: if $a : X \rightarrow A$ is finite and \mathcal{F} is CGG on X , then $a_* \mathcal{F}$ is CGG on A .
- (iv) The case $k = 2$ follows from the fact that ω_X is CGG away from Z and Proposition 61(i) (ω_X is a line bundle). For $k > 2$, one argues by induction.

□

We now provide a sketch of Pareschi and Popa's proof of Theorem 56 when $\chi(\omega_X) > 0$. As above, denote by Z the exceptional locus of $a : X \rightarrow A$.

Theorem 63. [34, Theorem 5.1] *Let Y be smooth projective variety of general type and maximal Albanese dimension. If the Albanese image of Y is not fibered by abelian subvarieties², then $\omega_Y^{\otimes 3}$ is very ample away from the exceptional locus Z of the Albanese map.*

Proof. We simply ennumerate the steps in the proof; see [34, Theorem 6.1]) for the details.

- (i) Showing that $\omega_Y^{\otimes 3}$ is very ample away from Z is equivalent to showing that $\mathcal{I}_y \otimes \omega_Y^{\otimes 3}$ is globally generated away from Z for every $y \in Y \setminus Z$.
- (ii) Since $a(Y)$ is not ruled by abelian subvarieties, then $a_* \omega_Y$ is M-regular, and ω_Y is hence CGG away from Z .
- (iii) The main issue in the proof is to show that $a_*(\mathcal{I}_y \otimes \omega_Y^2 \otimes \mathcal{J}(|\omega_Y|))$ is IT_0 , and hence M-regular and CGG. Therefore, $\mathcal{I}_y \otimes \omega_Y^2 \otimes \mathcal{J}(|\omega_Y|)$ is also CGG away from Z .

²This assumption is weaker than Chen-Hacon's hypotheses that $\chi(\omega_Y) > 0$. By the Green-Lazarsfeld vanishing theorem, $\chi(\omega_Y) \geq 0$ for varieties Y of maximal Albanese dimension. Ein and Lazarsfeld then show that if Y is of maximal Albanese dimension and $\chi(\omega_Y) = 0$, then the Albanese image is ruled by subtori of $\text{Alb}(Y)$.

- (iv) By (ii) and (iii), since both $\mathcal{I}_y \otimes \omega_Y^{\otimes 2} \otimes \mathcal{J}(\|\omega_Y\|)$ and ω_Y are CGG, then their product $\mathcal{I}_y \otimes \omega_Y^{\otimes 3} \otimes \mathcal{J}(\|\omega_Y\|)$ is globally generated away from Z , by Proposition 61(i).
- (v) The zero locus of $\mathcal{J}(\|\omega_Y\|)$ is contained³ in Z , it follows from (iv) that $\mathcal{I}_y \otimes \omega_Y^{\otimes 3}$ is globally generated away from Z as claimed.

□

5.1.2 Case $\chi(\omega_X) = 0$

To attack the general case, Jiang, Lahoz and Tirabassi implement in [24] an inductive argument based on the following Lemma:

Lemma 64. [24, Lemma 2.1] *Let $f : X \rightarrow Y$ be fibration between smooth projective varieties of general type and let $L \in \text{Pic}(X)$. Assume that the following three conditions hold:*

1. *The image of the restriction map $H^0(X, L) \rightarrow H^0(X, L \otimes k(y)) \simeq H^0(X_y, L_y)$ induces a birational map.*
2. *There are line bundles $\{H_i\}_{i=1, \dots, M} \subset \text{Pic}(Y)$ and a map*

$$\pi : \mathbf{P}(H^0(X, L)^\vee) \longrightarrow \mathbf{P}(H^0(Y, H_1)^\vee) \times \cdots \times \mathbf{P}(H^0(Y, H_M)^\vee)$$

such that:

2.1 The diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi|_L} & \mathbf{P}(H^0(X, L)^\vee) \\ f \downarrow & & \downarrow \pi \\ Y & \xrightarrow{\varphi_Y} & \mathbf{P}(H^0(Y, H_1)^\vee) \times \cdots \times \mathbf{P}(H^0(Y, H_M)^\vee) \end{array}$$

is commutative.

2.2 The multiple evaluation map

$$\varphi_Y : Y \longrightarrow \mathbb{P}(H^0(Y, H_1)^\vee) \times \cdots \times \mathbb{P}(H^0(Y, H_M)^\vee)$$

is birational

Then the linear system $|L|$ induces a birational map.

³Indeed, on the one hand, we have $\text{Zeros}(\mathcal{J}(\|\omega_Y\|)) \subseteq \text{Bs}(\omega_X^2 \otimes \alpha^2)$ for all $\alpha \in \hat{A}$, and on the other hand, $\text{Bs}(\omega_X^2 \otimes \alpha^2) \subset Z$ by Theorem 62(iv).

The argument in [24] goes along the following lines. Given a smooth projective variety of maximal Albanese dimension and general type, the birationality of φ_{3K_X+P} is proven in [24] using the above lemma by induction on the dimension. The curve case being classical, one may assume the statement to be true for all smooth projective varieties of maximal Albanese dimension and general type of dimension $\leq \dim X - 1$. The difficulty in studying the tricanonical map resides in the fact that the sheaf $a_{X,*}\omega_X$ may fail to be M-regular, this failure being measured by the set

$$\mathcal{J}_X = \{0 < j < \dim X, \quad V_{a_X}^j(\omega_X) \text{ has a component of codimension } j\}$$

If $\mathcal{J}_X = \emptyset$ (one then says that the sheaf $a_{X,*}\omega_X$ is *almost M-regular*⁴), since X is of general type, a result of Ein and Lazarsfeld (c.f. Theorem 3 in [10]) shows that $\chi(\omega_X) > 0$ and this case was handled in [9]. On the other hand, let k be the largest integer in \mathcal{J}_X , let $Q + \hat{B} \subseteq V_{a_X}^k(\omega_X)$ be a codimension- k component, where $Q \in \text{Pic}^0(X) \setminus \hat{B}$ is a torsion element, and consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{a_X} & A \\ f \downarrow & & \downarrow \pi \\ Y & \xrightarrow{b} & B \end{array}$$

where $f : X \rightarrow Y$ is a suitable birational modification of the Stein factorization of $\pi \circ a$ such that Y is smooth. The authors then show that for a general point $y \in Y$ and for all $P \in \text{Pic}^0(X)$, the restriction map

$$H^0(X, \mathcal{O}_X(3K_X) \otimes P) \twoheadrightarrow H^0(X_y, \mathcal{O}_{X_y}(3K_{X_y}) \otimes P)$$

is surjective, so by virtue of the induction hypothesis, its image induces a birational map, which shows that condition (1) in Lemma 64 is satisfied for $L = \mathcal{O}_X(3K_X) \otimes P$. The rest of the proof boils down to constructing suitable sheaves $H_1, \dots, H_M \in \text{Pic}(Y)$ which satisfy condition 2.1 and 2.2 in Lemma 64.

5.2 The 4th canonical map in positive characteristic [43]

The main result in [43] is a proof of the birationality of the fourth canonical map for varieties of general type and maximal Albanese dimension in positive characteristic.

⁴It is shown in [2] (c.f. Proposition 4.8) that if $\mathcal{J}_X = \emptyset$ and $V_{a_X}^d(\omega_X) = \{\hat{0}\}$, then the torsion of the sheaf $\widehat{R\Delta\omega_X}$ is isomorphic to $k(\hat{0})$.

Theorem 65. [43, Theorem 5.6] *Let Y be a smooth projective variety of general type and maximal Albanese dimension over an algebraically closed field of positive characteristic. If furthermore the Albanese map is separable, then $\omega_X^{\otimes 4}$ induces a birational map.*

Setup: Let $a : X \rightarrow A$ be a nontrivial morphism from a smooth projective variety of general type X to an abelian variety A . Since X is big, write $K_X \sim_{\mathbb{Q}} H + E$ with H ample and E effective. Write $\Delta = (1 - \epsilon)K_X + \epsilon E$ with $\epsilon = (0, 1) \cap \mathbb{Q}$. Fix l such that $l\Delta$ is Cartier. Set

$$\mathcal{F}_r = \mathcal{O}_X((r+1)K_X) \otimes \tau(|r\Delta|)$$

Set $\mathfrak{a}_m = bs(|ml\Delta|)$ and write $\tau(|r\Delta|) = \tau(\mathfrak{a}_{rm}^{1/ml}) = \left(\mathfrak{a}_{rm}^{[p^e/ml]}\right)^{[1/p^e]}$ for every $e \gg 0$. The iterated trace map gives

$$Tr^e : F_*^e \left(\mathfrak{a}_{rm}^{[p^e/ml]} \cdot \mathcal{O}_X(K_X) \right) \longrightarrow \tau(|r\Delta|) \cdot \mathcal{O}_X(K_X)$$

so tensoring with $\mathcal{O}_X(K_X)$ yields

$$Tr^e : F_*^e \underbrace{\left(\mathfrak{a}_{rm}^{[p^e/ml]} \cdot \mathcal{O}_X((rp^e + 1)K_X) \right)}_{\stackrel{\text{not}}{=} \tilde{\mathcal{F}}_{r,e}} \longrightarrow \tau(|r\Delta|) \cdot \mathcal{F}_r$$

It is easy to see that we also get a surjection $\mathcal{F}_{r,e} \twoheadrightarrow \tilde{\mathcal{F}}_{r,e}$ where

$$\mathcal{F}_{r,e} = V_{r,e} \otimes \mathcal{O}_X(-rml[p^e/ml]\Delta + (rp^e + 1)K_X)$$

so that in conclusion, we have two surjections

$$F_*^e \mathcal{F}_{r,e} \twoheadrightarrow F_*^e \tilde{\mathcal{F}}_{r,e} \twoheadrightarrow \mathcal{F}_r$$

The proof relies on the following vanishing theorems:

Lemma 66. [43, Lemma 4.1, Lemma 4.2]

- (i) *For any fixed $r > 0$, $R^i f_*(F_*^e \mathcal{F}_{r,e}) = 0$ for all $i > 0$ and all $e \gg 0$.*
- (ii) *For any fixed $r > 0$, $H^i(A, a_*(F_*^e \mathcal{F}_{r,e}) \otimes P) = 0$ for all $i > 0$, all $e \gg 0$ and all $P \in \hat{A}$.*

For any ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$, we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & (F_*^e \mathcal{F}_{r,e})_{\mathcal{I}} & \longrightarrow & F_*^e \mathcal{F}_{r,e} & \longrightarrow & \mathcal{F}_{r,e} \otimes \mathcal{O}_X/\mathcal{I} \longrightarrow 0 \\ & & \downarrow \phi_{r,e,\mathcal{I}} & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{F}_r \otimes \mathcal{I} & \longrightarrow & \mathcal{F}_r & \longrightarrow & \mathcal{F}_r \otimes \mathcal{O}_X/\mathcal{I} \longrightarrow 0 \end{array}$$

where the left vertical surjection follows from the 5-lemma. The main technical result employed in [43] is the following:

Lemma 67. [43, Theorem 5.2] Fix $e >$ and $r > 0$ and let $\mathcal{I} \subset \mathcal{O}_X$ be an ideal sheaf whose co-support does not meet that of $\tau(||r\Delta||)$. Suppose that $x \in X$ is a point such that:

1. x is not in the co-support of $\tau(||r\Delta||)$ or \mathcal{I} ,
2. $a_*(F_*^e \mathcal{F}_{r,e})_{\mathcal{I},x} \neq a_*(F_*^e \mathcal{F}_{r,e})_{\mathcal{I}}$,
3. $H^i(A, a_*(F_*^e \mathcal{F}_{r,e})_{\mathcal{I}} \otimes P) = 0$ for all $i > 0$ and all $P \in \hat{A}$.

Then the surjection $(F_*^e \mathcal{F}_{r,e})_{\mathcal{I}} \rightarrow \mathcal{F}_r \otimes k(x)$ induces a surjection

$$H^0(X, (F_*^e \mathcal{F}_{r,e})_{\mathcal{I}} \otimes a^*P) \rightarrow H^0(X, \mathcal{F}_r \otimes k(x) \otimes a^*P)$$

for general $P \in \hat{A}$. Moreover, since there is a factorization $(F_*^e \mathcal{F}_{r,e})_{\mathcal{I}} \rightarrow \mathcal{F}_r \otimes \mathcal{I} \rightarrow \mathcal{F}_r \otimes k(x)$, we also get a surjection

$$H^0(X, \mathcal{F}_r \otimes \mathcal{I} \otimes a^*P) \rightarrow H^0(X, \mathcal{F}_r \otimes k(x) \otimes a^*P)$$

so that x is not a base-point of $\mathcal{F}_r \otimes \mathcal{I} \otimes a^*P$ for general $P \in \hat{A}$.

We also record the following lemma for future reference:

Lemma 68. [43, Theorem 5.6, Step 3] Using the notation introduced above, is $x \in X$ is a general point (more concretely, a point which does not lie in the co-support of $\tau(||\Delta||)$), then the sheaf $a_*(F_*^e \mathcal{F}_{1,e})_{\mathcal{I}_x}$ satisfies IT_0 for every $e \gg 0$, namely

$$H^i\left(A, a_*(F_*^e \mathcal{F}_{1,e})_{\mathcal{I}_x} \otimes P\right) = 0, \quad \forall i > 0, \quad \forall P \in \hat{A}$$

5.3 Argument in positive characteristic

As in the characteristic zero case, we split the argument into two cases, which roughly correspond to $\chi(\omega_X) = 0$ and $\chi(\omega_X) > 0$.

5.3.1 Case in which $a(X)$ is fibered by abelian subvarieties

We start by proving the following statement.

Theorem 69. Let X be smooth projective variety of general type and maximal Albanese dimension. Denote by $a : X \rightarrow A$ the Albanese morphism and set $\Omega_e = F_*^e a_* \omega_X$, $\Lambda_e = R\hat{S}D_A \Omega_e$ and $\Lambda = \varinjlim \Lambda_e$. If $\{\Omega_e\}$ is M -regular (i.e. $\mathcal{H}^0(\Lambda)$ is torsion-free), then $\omega_Y^{\otimes 3}$ is very ample away from the exceptional locus Z of the Albanese map.

Remark 70. By Theorem 44, we may replace the assumption on the M -regularity of $\{\Omega_e\}$ by the assumption that the image of $a : X \rightarrow A$ is not ruled by abelian subvarieties of A .

Our first step is to prove a positive characteristic analogue of the implication (iii) \Rightarrow (iv) in Lemma 59.

Lemma 71. *If $\{\Omega_e\}$ is an M -regular inverse system on an abelian variety A , in the sense that $\mathcal{H}^0(\Lambda)$ is a torsion-free sheaf, then for any open subset $U \subset \hat{A}$ and every sheaf H on X satisfying IT_0 , the sum of multiplication maps of global sections*

$$\mathcal{M}_U : \bigoplus_{\alpha \in \hat{A}} H^0(A, \varprojlim_e \Omega_e \otimes P_\alpha) \otimes H^0(A, H \otimes P_\alpha^\vee) \xrightarrow{\oplus m_\alpha} H^0(A, \varprojlim_e \Omega_e \otimes H)$$

is surjective.

Remark 72. *We will be applying this to the sheaf $H = a_*(F_*^{e'} \mathcal{F}_{1,e'})$, which satisfies IT_0 for $e' \gg 0$ by Lemma 68.*

Proof. If $\{\Omega_e\}$ is an M -regular inverse system, letting $\Lambda_e = R\hat{S}D_A\Omega_e$ and $\Lambda = \varinjlim \Lambda_e$ we have that $\mathcal{H}^0(\Lambda)$ is torsion-free (by definition) and we likewise have an injection

$$Hom(R^0\hat{S}(H), \mathcal{H}^0(\Lambda)) \hookrightarrow \prod_{\alpha \in U} Hom(R^0\hat{S}(H), \mathcal{H}^0(\Lambda)) \otimes k(\alpha) \quad (5.1)$$

Regarding the left-hand side of (5.1), note that

$$\begin{aligned} Hom(R^0\hat{S}(H), \mathcal{H}^0(\Lambda)) &= Hom(R\hat{S}(H), \Lambda) \simeq \varinjlim Hom(R\hat{S}(H), \Lambda_e) \\ &\stackrel{\text{Mukai}}{=} \varinjlim Ext^g(H, D_A\Omega_e) \\ &\stackrel{SD}{=} \varinjlim D_k Hom(D_A\Omega_e, H) \end{aligned}$$

so dualizing, we obtain

$$\begin{aligned} D_k Hom(R^0\hat{S}(H), \mathcal{H}^0(\Lambda)) &= D_k \varinjlim D_k Hom(D_A\Omega_e, H) = \varprojlim Hom(D_A\Omega_e, H) \\ &= \varprojlim Hom(\mathcal{O}_A, \Omega_e \otimes H) = \varprojlim H^0(A, \Omega_e \otimes H) \end{aligned}$$

As for the right-hand side of (5.1), we have the following computation for every Ω_e :

$$\begin{aligned} Hom(R^0\hat{S}(H), R^0\hat{S}(D_A(\Omega_e))) \otimes k(\alpha) &\simeq \mathcal{H}om(\mathcal{O}_{\hat{A}}, (R^0\hat{S}(H))^\vee \otimes R^0\hat{S}(D_A(\Omega_e))) \otimes k(\alpha) \\ &\simeq (R^0\hat{S}(H))^\vee \otimes R^0\hat{S}(D_A(\Omega_e)) \otimes k(\alpha) \\ &\simeq (R^0\hat{S}(H))^\vee \otimes \mathcal{H}^0(\Lambda_e) \otimes k(\alpha) \\ &\simeq H^0(A, \Omega_e \otimes P_\alpha^\vee)^\vee \otimes H^0(A, H \otimes P_\alpha)^\vee \end{aligned}$$

where the first isomorphism follows from the fact that $R^0\hat{S}(H)$ is locally free, the second one is obvious, the third one is the definition of Λ_e and the last one follows from Corollary 26[i)]. Taking direct limits, we obtain

$$\varinjlim_e (H^0(A, \Omega_e \otimes P_\alpha)^\vee \otimes H^0(A, H \otimes P_\alpha^\vee)^\vee) \simeq Hom(R^0\hat{S}(H), \mathcal{H}^0(\Lambda)) \otimes k(\alpha)$$

and we conclude that the torsion-freeness of $\mathcal{H}^0(\Lambda)$ implies the surjectivity of

$$\mathcal{M}_U : \bigoplus_{\alpha \in \hat{A}} H^0(A, \varprojlim_e \Omega_e \otimes P_\alpha) \otimes H^0(A, H \otimes P_\alpha^\vee) \xrightarrow{\oplus m_\alpha} H^0(A, \varprojlim_e \Omega_e \otimes H)$$

for every open subset $U \subset \hat{A}$. \square

Remark 73. *If the sheaf $\mathcal{H}^0(\Lambda)$ has torsion supported on points, the conclusion of Lemma 71 remains true for any open subset which contains those torsion points. Indeed, the kernel of the evaluation map (5.1) generates a torsion subsheaf of $\mathcal{H}^0(\Lambda) \otimes R^0\hat{S}(H)^\vee$ supported on the complement of U , so the map (5.1) remains injective provided that there is no torsion outside of U .*

In particular, if the system $\{\Omega_e\}$ is Mittag-leffler, we have

$$H^0(A, \varprojlim_e \Omega_e \otimes H) \simeq \varprojlim_e H^0(A, \Omega_e \otimes H) \twoheadrightarrow S^0(\Omega_0 \otimes H)$$

and the above surjectivity implies that the subspaces $\{Im(m_\alpha)\}_{\alpha \in U}$ span $H^0(A, \varprojlim_e \Omega_e \otimes H)$, and hence also $S^0(\Omega_0 \otimes H)$, which is finitely generated, so there exists an integer N such that N general subspaces $\{Im(m_{\alpha_i})\}_{i=1, \dots, N}$ span $S^0(\Omega_0 \otimes H)$. Therefore, there is a surjection

$$\bigoplus_{i=1}^N H^0(A, \varprojlim_e \Omega_e \otimes P_{\alpha_i}) \otimes H^0(A, H \otimes P_{\alpha_i}^\vee) \xrightarrow{\oplus m_{\alpha_i}} S^0(A, \Omega_0 \otimes H).$$

This motivates the following adapted definition of continuous global generation:

Definition 74. *Let $\{\Omega_e\}$ be an inverse system of coherent sheaves on an abelian variety A which satisfies the Mittag-Leffler condition. We define one such system to be continuously globally generated (CGG for short) if for any nonempty open subset $U \subset \hat{A}$, the sum of evaluation maps*

$$\bigoplus_{\alpha \in U} H^0(A, \varprojlim_e \Omega_e \otimes P_\alpha) \otimes P_\alpha^\vee \longrightarrow \varprojlim_e \Omega_e$$

is surjective.

In particular we can show that M-regularity implies continuous global generation

Corollary 75. *If $\{\Omega_e\}$ is an M-regular system (i.e. $\Lambda = \varinjlim_e R\hat{S}D_A(\Omega_e)$ is torsion-free) on an abelian variety A satisfying the Mittag-Leffler condition, there exists an integer N such that for general $\alpha_1, \dots, \alpha_N \in \hat{A}$, the sum of twisted evaluation maps*

$$\bigoplus_{i=1}^N H^0(A, \varprojlim_e \Omega_e \otimes P_{\alpha_i}) \otimes P_{\alpha_i}^\vee \longrightarrow S^0(\Omega_0)$$

is surjective.

We now proceed to prove Theorem 69.

Proof. Since the system $\{F_*^e a_* \omega_X\}$ is M-regular and the sheaf $a_*(F_*^{e'} \mathcal{F}_{1,e'})$ is IT_0 , we may consider the following commutative diagram

$$\begin{array}{ccc}
\oplus_{\alpha \in U} H^0(a_*(F_*^{e'} \mathcal{F}_{1,e'}) \otimes P_\alpha) \otimes H^0(\varprojlim_e F_*^e a_* \omega_X \otimes P_\alpha^\vee) \otimes \mathcal{O}_A & \xrightarrow{[1]} & H^0(\varprojlim_e F_*^e a_* \omega_X \otimes a_*(F_*^{e'} \mathcal{F}_{1,e'})) \otimes \mathcal{O}_A \\
\downarrow ev & & \downarrow ev \\
\oplus_{\alpha \in U} H^0(a_*(F_*^{e'} \mathcal{F}_{1,e'}) \otimes P_\alpha) \otimes (P_\alpha^\vee \otimes \varprojlim_e F_*^e a_* \omega_X) & \xrightarrow{[4]} & a_*(F_*^{e'} \mathcal{F}_{1,e'}) \otimes \varprojlim_e F_*^e a_* \omega_X \\
\downarrow & & \downarrow [3] \\
\oplus_{\alpha \in U} H^0(a_*(F_*^{e'} \mathcal{F}_{1,e'}) \otimes P_\alpha) \otimes P_\alpha^\vee \otimes a_* \omega_X & \xrightarrow{[2]} & a_*(F_*^{e'} \mathcal{F}_{1,e'}) \otimes S^0 a_* \omega_X
\end{array}$$

where

- (i) The surjection in [1] follows from the fact that $\{F_*^e a_* \omega_X\}$ is M-regular away from $Z = Exc(a)$ and $(F_*^{e'} \mathcal{F}_{1,e'})$ is IT_0 for $e' \gg 0$.
- (ii) The surjection in [2] follows from the fact that $a_*(F_*^{e'} \mathcal{F}_{1,e'})$ is CGG (being IT_0).
- (iii) The surjection in [3] follows from the fact that $\varprojlim_e F_*^e a_* \omega_X \rightarrow S^0 a_* \omega_X$ was surjective, by definition.
- (iv) The surjection in [4] follows from the fact that the map

$$H^0(X, a_*(F_*^{e'} \mathcal{F}_{1,e'}) \otimes P_\alpha) \otimes P_\alpha^\vee \rightarrow a_*(F_*^{e'} \mathcal{F}_{1,e'})$$

is surjective, $a_*(F_*^{e'} \mathcal{F}_{1,e'})$ being IT_0 .

It follows that we have a surjection

$$H^0\left(A, \varprojlim_e F_*^e a_* \omega_X \otimes a_*(F_*^{e'} \mathcal{F}_{1,e'})\right) \otimes \mathcal{O}_A \rightarrow a_*(F_*^{e'} \mathcal{F}_{1,e'}) \otimes S^0 a_* \omega_X$$

If we could show that there is actually a surjection

$$H^0\left(A, \varprojlim_e F_*^e a_* \omega_X \otimes a_*(F_*^{e'} \mathcal{F}_{1,e'})\right) \otimes \mathcal{O}_A \rightarrow a_*(F_*^{e'} \mathcal{F}_{1,e'}) \otimes a_* \omega_X$$

then we would have a surjection

$$H^0\left(X, \varprojlim_e F_*^e \omega_X \otimes F_*^{e'} \mathcal{F}_{1,e'}\right) \otimes \mathcal{O}_A \rightarrow F_*^{e'} \mathcal{F}_{1,e'} \otimes \omega_X$$

away from the exceptional locus, which yields a commutative diagram

$$\begin{array}{ccccc}
 H^0(X, \varprojlim F_*^e \omega_X \otimes F_*^{e'} \mathcal{F}_{1,e'}) \otimes \mathcal{O}_X & \twoheadrightarrow & F_*^{e'} \mathcal{F}_{1,e'} \otimes \omega_X & \twoheadrightarrow & \mathcal{F}_1 \otimes \omega_X \\
 & \searrow & \uparrow & & \uparrow [*] \\
 & & H^0(X, F_*^{e'} \mathcal{F}_{1,e'} \otimes \omega_X) \otimes \mathcal{O}_X & \longrightarrow & H^0(X, \omega_X \otimes \mathcal{F}_1) \cdot \mathcal{O}_X
 \end{array}$$

We thus conclude that the map $[\ast]$ is surjective away from the exceptional locus of a , and hence that the sheaf

$$\omega_X \otimes \mathcal{F}_1 = \omega_X^3 \otimes \tau(\|\Delta\|)$$

is globally generated away from $\text{Exc}(a)$, as claimed. \square

5.3.2 General case

In order to tackle the general case, we seek to apply Lemma 64 above to the sheaves $L = \mathcal{F}_2 \otimes P$ and $H_i = \mathcal{F}_2^Y \otimes Q_i$, where $P \in \text{Pic}^0(X)$, $Q_i \in \text{Pic}^0(Y)$. The sheaves \mathcal{F}_2 and \mathcal{F}_e^Y have been described in section 5.2 (note that we need both X and Y to be of general type) to construct them.

Recall that given the Cartier module $\{\Omega_e = F_*^e a_* \omega_X\}_e$, by Hacon and Patakfalvi's Theorem 25, we have that the complex $\Lambda = \varinjlim RS_{A,\hat{A}} D_A(\Omega_e)$ is actually a quasi-coherent sheaf concentrated in degree 0, namely $\Lambda \simeq \mathcal{H}^0(\Lambda)$, and by Theorem 39, we also know that the torsion-freeness of this sheaf provides a reasonable notion of M-regularity.

We propose to mimic the approach in [24] by measuring the failure of the sheaf $\mathcal{H}^0(\Lambda)$ to be torsion-free via the set

$$\mathcal{J}_X = \{0 < j < \dim X, \quad \mathcal{H}^0(\Lambda) \text{ has a torsion point of dimension } j\}.$$

In particular, one sees that if \mathcal{J}_X is empty, then the torsion $\mathcal{H}^0(\Lambda)$ must be supported in dimension 0 so we may conclude by Theorem 71 and Remark 73.

If on the other hand we have that $\mathcal{J}_X \neq \emptyset$, namely the sheaf $\mathcal{H}^0(\Lambda)$ has torsion in strictly positive dimension, and $w \in \hat{A}$ is a torsion point of maximal dimension, then $\hat{B} := \overline{\{w\}} \subset \hat{A}$ is a torsion translate of an abelian subvariety of \hat{A} . We assume that \hat{B} is a translate through the origin for simplicity. Let $\dim \hat{W} = k$ and consider the following commutative diagram

$$\begin{array}{ccc}
 X & \xrightarrow{a_X} & A_X \\
 f \downarrow & & \downarrow pr \\
 Y & \xrightarrow{a_Y} & B
 \end{array}$$

where $f : X \rightarrow Y$ is the Stein factorization of $pr \circ a_X : X \rightarrow B$. Assume for the moment that Y is smooth (and see 5.3.2.3 for comments on the nonsmooth case). Consider the

inverse system $\{\Omega_{Y,e} = F_*^e b_{Y,*} \omega_Y\}_e$ and the corresponding quasi-coherent sheaf $\Lambda_Y = \varinjlim_e RS_{B,\hat{B}} D_B(\Omega_{Y,e})$.

5.3.2.1 Case in which $\mathcal{H}^0(\Lambda_Y)$ is torsion-free

In this case, we have that Y is of general type: indeed, if it was not, the Itaka fibration $Y \rightarrow Z$ would be fibered by abelian subvarieties (its fibers F have $\kappa_S(F) = 0$ and maximal Albanese dimension, so they are abelian by Theorem 1.1.1 in [15]) and therefore, $H^0(Y, \omega_F \otimes P) = 0$ for all $P \in \hat{B}$. By adjunction, we have $H^0(Y, \omega_Y \otimes P) = 0$ and hence $S^0(\omega_Y \otimes P) := \bigcap_{e \geq 0} \text{Im}(H^0(Y, F_*^e \omega_Y \otimes P) \rightarrow H^0(Y, \omega_Y \otimes P)) = 0$. By Corollary 3.2.1 in [15], it follows that $rk \mathcal{H}^0(\Lambda_Y) = h^0(\varprojlim_e \Omega_{Y,e} \otimes P) = 0$, and hence that $\mathcal{H}^0(\Lambda_Y)$ is a torsion sheaf, a contradiction.

We now seek to use Lemma 64 with $L = \mathcal{F}_2 \otimes P$ and $H_i = \mathcal{F}_2^Y \otimes Q_i$, where $P \in \text{Pic}^0(X)$, $Q_i \in \text{Pic}^0(Y)$, in order to show that the sheaf $\mathcal{F}_2 \otimes a_X^* P$ separates general points of X for all $P \in \text{Pic}^0(A)$. We thus need to show the following items:

- (i) For general $y \in Y$, the restriction map

$$H^0(X, \mathcal{F}_2 \otimes a^* P) \rightarrow H^0(X_y, \mathcal{F}_{2|X_y} \otimes a^* P)$$

is surjective.

- (ii) There exist $Q_1, \dots, Q_M \in \text{Pic}^0(Y)$ such that the multiple evaluation map

$$\varphi_{Q_1, \dots, Q_M} : Y \rightarrow \mathbf{P}(H^0(Y, \mathcal{F}_2^Y \otimes Q_1)^\vee) \times \dots \times \mathbf{P}(H^0(Y, \mathcal{F}_2^Y \otimes Q_M)^\vee)$$

is birational.

- (iii) For all $P \in \text{Pic}^0(A)$ and all $Q \in \text{Pic}^0(B)$, there are maps

$$\mathbf{P}(H^0(X, \mathcal{F}_2 \otimes P)^\vee) \rightarrow \mathbf{P}(H^0(Y, \mathcal{F}_2^Y \otimes P)^\vee)$$

such that the diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathbf{P}(H^0(X, \mathcal{F}_2 \otimes P)^\vee) \\ f \downarrow & & \downarrow \\ Y & \longrightarrow & \mathbf{P}(H^0(Y, \mathcal{F}_2^Y \otimes Q)^\vee) \end{array}$$

commutes.

So far, we have been able to proof items (ii) and (iii) (see Lemmas 78 and 79 below). With regards to item one, all we have been able to prove thus far is an analogous surjection for the vector bundle $F_*^e \mathcal{F}_{2,e}$, as the following lemma shows.

Lemma 76. *With the same notations as above, for general $y \in Y$, the restriction map*

$$H^0(X, F_*^e \mathcal{F}_{2,2} \otimes a^* P) \rightarrow H^0(X_y, F_*^e \mathcal{F}_{2,e|X_y} \otimes a^* P)$$

is surjective.

Proof. We prove the statement for $P = \mathcal{O}_X$. With the notation introduced in section 5.2, consider the sheaf

$$\tilde{\mathcal{F}}_{2,e} = V_{2,e} \otimes \mathcal{O}_X (-2ml[p^e/ml]\Delta + (2p^e + 1)K_X - p^e f^* K_Y).$$

By Lemma 4.2 in [43], we know that the sheaf $F_*^e \tilde{\mathcal{F}}_{2,e}$ is IT^0 , so in particular it is CGG (in the usual sense of [32]) and there is a finite set of elements $P_\alpha \in \text{Pic}^0(X)$ such that

$$\bigoplus_{\alpha=1}^N H^0(X, F_*^e \tilde{\mathcal{F}}_{2,e} \otimes P_\alpha) \otimes P_\alpha^\vee \rightarrow F_*^e \tilde{\mathcal{F}}_{2,e}.$$

Consider the following commutative diagram, obtained by alternating the order of the obvious multiplication and evaluation maps

$$\begin{array}{ccc} \bigoplus_{\alpha} H^0(X, F_*^e \tilde{\mathcal{F}}_{2,e} \otimes P_\alpha) \otimes H^0(X, f^* \omega_Y \otimes P_\alpha^\vee) & \longrightarrow & H^0(X, F_*^e \tilde{\mathcal{F}}_{2,e} \otimes f^* \omega_Y) \otimes \mathcal{O}_Y \\ \downarrow & & \downarrow \\ \bigoplus_{\alpha} H^0(X, F_*^e \tilde{\mathcal{F}}_{2,e} \otimes P_\alpha) \otimes P_\alpha^\vee \otimes f^* \omega_Y & \longrightarrow & F_*^e \tilde{\mathcal{F}}_{2,e} \otimes f^* \omega_Y \end{array}$$

Note that the bottom horizontal map is surjective by our choice of $\{P_\alpha\}_{\alpha=1}^N$ and the left vertical map is surjective over $X \setminus Bs(f^* \omega_Y)$, so the bottom left composition is surjective over an open subset of Y and hence, so is the evaluation map

$$H^0(X, F_*^e \tilde{\mathcal{F}}_{2,e} \otimes f^* \omega_Y) \rightarrow F_*^e \tilde{\mathcal{F}}_{2,e} \otimes f^* \omega_Y.$$

Note, however, that

$$F_*^e \tilde{\mathcal{F}}_{2,e} \otimes f^* \omega_Y \simeq F_*^e \mathcal{F}_{2,e}$$

so we generically have a surjection

$$H^0(X, F_*^e \mathcal{F}_{2,e}) \rightarrow F_*^e \mathcal{F}_{2,e}.$$

Composing with the evaluation map at a general $y \in Y$ and applying the cohomology and base change theorem, we obtain that

$$H^0(X, F_*^e \mathcal{F}_{2,e}) \simeq H^0(Y, f_*(F_*^e \mathcal{F}_{2,e})) \twoheadrightarrow f_*(F_*^e \mathcal{F}_{2,e}) \twoheadrightarrow f_*(F_*^e \mathcal{F}_{2,e}) \otimes k(y) \simeq H^0(X_y, (F_*^e \mathcal{F}_{2,e})|_{X_y})$$

The same argument also shows that actually for every $P \in \text{Pic}^0(X)$ and for general $y \in Y$, there is a surjection

$$H^0(X, F_*^e \mathcal{F}_{2,e} \otimes a^* P) \twoheadrightarrow H^0(X_y, (F_*^e \mathcal{F}_{2,e})|_{X_y} \otimes a^* P)$$

as claimed. \square

Remark 77. Note that the surjection $F_*^e \mathcal{F}_{2,e} \twoheadrightarrow \mathcal{F}_2$ and the above restriction maps induce a commutative diagram

$$\begin{array}{ccc} H^0(X, F_*^e \mathcal{F}_{2,e} \otimes a^* P) & \twoheadrightarrow & H^0(X_y, (F_*^e \mathcal{F}_{2,e})|_{X_y} \otimes a^* P) & \simeq f_*(F_*^e \mathcal{F}_{2,e}) \otimes b^* P \otimes k(y) \\ \downarrow & & \downarrow [*] & \\ H^0(X, \mathcal{F}_2 \otimes a^* P) & \longrightarrow & H^0(X_y, \mathcal{F}_2|_{X_y} \otimes a^* P) & \simeq f_*(\mathcal{F}_2) \otimes b^* P \otimes k(y) \end{array}$$

If the map $[*]$ was surjective, we would have the surjection that we need. Unfortunately that need not be the case, since that map is induced by $f_*(F_*^e \mathcal{F}_{2,e}) \rightarrow f_* \mathcal{F}_2$, which has a cokernel $R^1 f_*(\ker[F_*^e \mathcal{F}_{2,e} \twoheadrightarrow \mathcal{F}_2])$. As discussed in section 5.2 however, the surjection $F_*^e \mathcal{F}_{2,e} \twoheadrightarrow \mathcal{F}_2$ is induced by the Frobenius trace map, so one would get a surjection in $[*]$ by replacing the sheaf $f_* \mathcal{F}_2$ by $S^0 f_* F_*^e \mathcal{F}_{2,e}$. One would then have to make sure that the rest of the argument holds for sheaves like these.

Lemma 78 (verification of condition 2.2 of Lemma 64). *With the same notations introduced above, there exist $Q_1, \dots, Q_M \in \text{Pic}^0(Y)$ such that the multiple evaluation map*

$$\varphi_{Q_1, \dots, Q_M} : Y \longrightarrow \mathbf{P}(H^0(Y, \mathcal{F}_2^Y \otimes Q_1)^\vee) \times \dots \times \mathbf{P}(H^0(Y, \mathcal{F}_2^Y \otimes Q_M)^\vee)$$

is birational.

Proof. Since we are assuming that Y is of general type, we may consider the sheaves $F_{*,e}^Y \mathcal{F}_r^Y \twoheadrightarrow \mathcal{F}_r^Y$.

We start by showing that for general $y, z \in Y$, the sheaf $F_*^e \mathcal{F}_2^Y \otimes \mathcal{I}_y$ is continuously globally generated on z . It is proven in Step 3 within the proof of Theorem 5.6 in [43] that the sheaf $a_*(F_*^e \mathcal{F}_{2,e}^Y)_{\mathcal{I}_y}$ is IT_0 provided that y is not in the co-support of the ideal $\tau(|\Delta|)$. It follows that $a_*(F_*^e \mathcal{F}_{2,e}^Y)_{\mathcal{I}_y}$ is M-regular, and hence CGG, so in particular $(F_*^e \mathcal{F}_{2,e}^Y)_{\mathcal{I}_y}$ is CGG outside $\text{Exc}(a_Y)$, namely for any $U \subset \text{Pic}^0(Y)$, there is a surjection

$$\bigoplus_{\alpha \in U} H^0(Y, (F_*^e \mathcal{F}_2^Y)_{\mathcal{I}_y} \otimes P_\alpha) \otimes P_\alpha^\vee \twoheadrightarrow (F_*^e \mathcal{F}_2^Y)_{\mathcal{I}_y}.$$

It is clear from the diagram below that this implies that the sheaf $\mathcal{F}_2^Y \otimes \mathcal{I}_y$ is also CGG.

$$\begin{array}{ccc} \bigoplus_{\alpha \in U} H^0(Y, (F_*^e \mathcal{F}_{2,e}^Y)_{\mathcal{I}_y} \otimes P_\alpha) \otimes P_\alpha^\vee & \twoheadrightarrow & (F_*^e \mathcal{F}_{2,e}^Y)_{\mathcal{I}_y} \\ \downarrow & & \downarrow \\ \bigoplus_{\alpha \in U} H^0(Y, \mathcal{F}_2^Y \otimes \mathcal{I}_y \otimes P_\alpha) \otimes P_\alpha^\vee & \longrightarrow & \mathcal{F}_2^Y \otimes \mathcal{I}_y \end{array}$$

In summary, we have that for any y not in the co-support of $\tau(|\Delta|)$ and any $z \notin \text{Exc}(a_Y)$ different from y , there exists $P \in \text{Pic}^0(Y)$ such that $\mathcal{F}_2^Y \otimes \mathcal{I}_y \otimes P$ is globally generated on z . Therefore, for any two different points $y, z \in Y \setminus (\text{cosupp}(\tau(|\Delta|)) \cup \text{Exc}(a_Y))$, there exists $P \in \text{Pic}^0(Y)$ and a divisor $D_P \in |\mathcal{F}_2^Y \otimes \mathcal{I}_y \otimes P|$ such that $y \in D_P$ and $z \notin D_P$, so that $\varphi_P(y) \neq \varphi_P(z)$. Taking P_1, \dots, P_M so that $\varphi_{P_1 \dots P_M}$ becomes stable, we conclude that $\varphi_{P_1 \dots P_M}$ is birational. \square

Lemma 79 (verification of condition 2.1 of Lemma 64). *With the same notations introduced above, For all $P \in \text{Pic}^0(A)$ and all $Q \in \text{Pic}^0(B)$ there are maps*

$$P(H^0(X, \mathcal{F}_2 \otimes P)^\vee) \rightarrow P(H^0(Y, \mathcal{F}_2^Y \otimes P)^\vee)$$

such that the diagram

$$\begin{array}{ccc} X & \longrightarrow & P(H^0(X, \mathcal{F}_2 \otimes P)^\vee) \\ f \downarrow & & \downarrow \\ Y & \longrightarrow & P(H^0(Y, \mathcal{F}_2^Y \otimes Q)^\vee) \end{array}$$

commutes.

Proof. We start by showing that

$$H^0\left(Y, f_*(F_*^e \mathcal{F}_{2,e} \otimes a_X^* P) \otimes F_*^e \mathcal{F}_{2,e}^{Y,\vee} \otimes a_Y^* Q^\vee\right) \neq 0, \quad \forall P \in \text{Pic}^0(A), \quad \forall Q \in \text{Pic}^0(B)$$

In step 2 within the proof of Theorem 5.6 in [43], Zhang shows that for general $y \in Y$ and for all $Q \in \text{Pic}^0(B)$, there is a surjective map

$$H^0(Y, F_*^e \mathcal{F}_{2,e}^Y \otimes a_Y^* Q) \rightarrow H^0(Y, \mathcal{F}_2^Y \otimes a_Y^* Q \otimes k(y))$$

so in particular $H^0(Y, F_*^e \mathcal{F}_{2,e}^Y \otimes a_Y^* Q) \neq 0$ for all $Q \in \text{Pic}^0(B)$, whence $V_{a_Y}^0(F_*^e \mathcal{F}_{2,e}^Y) = \hat{B}$ and we are thus reduced to showing that

$$H^0\left(Y, f_*(F_*^e \mathcal{F}_{2,e} \otimes a_X^* P) \otimes F_*^e \mathcal{F}_{2,e}^{Y,\vee}\right) \neq 0, \quad \forall P \in \text{Pic}^0(A).$$

By Lemma 4.1 in [43], we have that $R^i f_*(F_*^e \mathcal{F}_{2,e}) = 0$ for all $i > 0$ and all $e \gg 0$, so in particular

$$R^i f_*\left(F_*^e \mathcal{F}_{2,e} \otimes a_X^* P \otimes f^* F_*^e \mathcal{F}_{2,e}^{Y,\vee}\right) = 0, \quad \forall i > 0, \quad \forall e \gg 0$$

so

$$\chi \left(Y, f_* \left(F_*^e \mathcal{F}_{2,e} \otimes a_X^* P \otimes f^* F_*^e \mathcal{F}_{2,e}^{Y,\vee} \right) \right) = \chi \left(X, F_*^e \mathcal{F}_{2,e} \otimes f^* F_*^e \mathcal{F}_{2,e}^{Y,\vee} \otimes a_X^* P \right)$$

is independent of $P \in \text{Pic}^0(A)$. Arguing as in the proof of Lemma 4.2 in [43], one sees that

$$H^i \left(X, F_*^e \mathcal{F}_{2,e} \otimes a_X^* P \otimes f^* F_*^e \mathcal{F}_{2,e}^{Y,\vee} \right) = 0, \quad \forall i > 0, \quad \forall e \gg 0$$

so

$$\begin{aligned} H^0 \left(X, F_*^e \mathcal{F}_{2,e} \otimes f^* (F_*^e \mathcal{F}_{2,e}^{Y,\vee}) \right) &= \chi \left(X, F_*^e \mathcal{F}_{2,e} \otimes f^* (F_*^e \mathcal{F}_{2,e}^{Y,\vee}) \right) \\ &= \chi \left(X, F_*^e \mathcal{F}_{2,e} \otimes f^* (F_*^e \mathcal{F}_{2,e}^{Y,\vee}) \otimes a_X^* P \right) = H^0 \left(X, F_*^e \mathcal{F}_{2,e} \otimes f^* (F_*^e \mathcal{F}_{2,e}^{Y,\vee}) \otimes a_X^* P \right) \end{aligned}$$

for any $P \in \text{Pic}^0(A)$, and we are thus reduced to showing that

$$H^0 \left(X, F_*^e \mathcal{F}_{2,e} \otimes f^* (F_*^e \mathcal{F}_{2,e}^{Y,\vee}) \right) \neq 0.$$

Recall that by definition

$$\mathcal{F}_{2,e} = V_{2,e} \otimes \mathcal{O}_X (-2ml[p^e/ml]\Delta_X + (2p^e + 1)K_X)$$

where $V_{2,e} = \text{Sym}^{[p^e/ml]} H^0(X, \mathcal{O}_X(2ml\Delta_X))$, $\Delta_X = (1-\epsilon)K_X + \epsilon E_X$ for $0 < \epsilon < 1$ rational and $K_X \simeq_{\mathbb{Q}} H_X + E_X$ with H_X ample and E_X effective (since K_X is big by assumption, we may choose H_X as positive as we want, and then pick E_X effective accordingly). We have the same definitions for $\mathcal{F}_{2,e}^Y$. Omitting the vector space $V_{2,e}$, we have

$$\mathcal{F}_{2,e} \otimes f^* \mathcal{F}_{2,e}^Y = \mathcal{O}_X \left((2p^e + 1) \underbrace{(K_X - f^* K_Y)}_{>0} + 2ml[p^e/ml](f^* \Delta_Y - \Delta_X) \right)$$

But note that

$$\begin{aligned} f^* \Delta_Y - \Delta_X &= (1-\epsilon)(f^* K_Y - K_X) + \epsilon(f^* E_Y - E_X) \\ &= (f^* K_Y - K_X) - \epsilon f^*(K_Y - E_Y) + \epsilon(K_X - E_X) = \underbrace{(f^* K_Y - K_X) - \epsilon f^* H_Y}_{<0} + \epsilon H_X \end{aligned}$$

so choosing H_X sufficiently ample in the decomposition of K_X , our claim follows.

We have thus shown that

$$H^0 \left(Y, f_*(F_*^e \mathcal{F}_{2,e} \otimes a_X^* P) \otimes F_*^e \mathcal{F}_{2,e}^{Y,\vee} \otimes a_Y^* Q^\vee \right) \neq 0, \quad \forall P \in \text{Pic}^0(A), \quad \forall Q \in \text{Pic}^0(B)$$

and we now seek to use these sections to show that for all $P \in \text{Pic}^0(A)$ and all $Q \in \text{Pic}^0(B)$, there are maps

$$\mathbf{P}(H^0(X, \mathcal{F}_2 \otimes P)^\vee) \rightarrow \mathbf{P}(H^0(Y, \mathcal{F}_2^Y \otimes Q)^\vee).$$

For fixed $P \in \text{Pic}^0(A)$ and $Q \in \text{Pic}^0(B)$, choose $y \in Y$ general enough so that the map

$$H^0(Y, F_*^e \mathcal{F}_{2,e}^Y \otimes Q) \rightarrow H^0(Y, \mathcal{F}_2^Y \otimes Q \otimes k(y))$$

is surjective⁵ and choose $s \in H^0 \left(Y, f_*(F_*^e \mathcal{F}_{2,e} \otimes a_X^* P) \otimes F_*^e \mathcal{F}_{2,e}^{Y,\vee} \otimes a_Y^* Q^\vee \right)$. Then consider the following commutative diagram

$$\begin{array}{ccc} H^0(Y, F_*^e \mathcal{F}_{2,e}^Y \otimes Q) & \xrightarrow{\cdot s} & H^0(Y, f_*(F_*^e \mathcal{F}_{2,e} \otimes P)) \\ \downarrow & & \downarrow \\ H^0(Y, \mathcal{F}_2^Y \otimes Q) & \cdots \cdots \cdots \rightarrow & H^0(Y, f_*(\mathcal{F}_2 \otimes P)) \\ \downarrow & & \downarrow \\ H^0(Y, \mathcal{F}_2^Y \otimes Q \otimes k(y)) & \xrightarrow{\cong} & H^0(Y, f_*(\mathcal{F}_2 \otimes P) \otimes k(y)) \end{array}$$

This diagram defines our sought for map

$$H^0(Y, \mathcal{F}_2^Y \otimes Q) \rightarrow H^0(Y, f_*(\mathcal{F}_2 \otimes P)).$$

□

If condition 1 of Lemma 64 could be verified entirely (c.f. Lemma 76 and Remark 77), one would conclude as in [24]. Arguing by induction on $n = \dim X$, we may assume that for any $P_Y \in \text{Pic}^0(Y)$, the sheaf $\mathcal{F}_2^Y \otimes P_Y$ induces a birational map for any smooth projective subvariety Y of maximal Albanese dimension, general type and $\dim Y \leq n - 1$. We may also assume that $\mathcal{H}^0(\Lambda)$ has torsion (or else we are done by the previous subsection), so there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{a_X} & A_X \\ f \downarrow & & \downarrow pr \\ Y & \xrightarrow{a_Y} & B \end{array}$$

as above, where $f : X \rightarrow Y$ is the Stein factorization of $pr \circ a_X : X \rightarrow B$.

⁵Again, this is the conclusion of Step 2 within the proof of Theorem 5.6 in [43].

We have seen that for general $y \in Y$, if X_y denotes a general fibre of f , the restriction map

$$H^0(X, \mathcal{F}_2 \otimes a_X^* P) \rightarrow H^0(X, \mathcal{F}_2 \otimes a_X^* P \otimes k(y)) \simeq H^0(X_y, \mathcal{F}_2 \otimes a_X^* P|_{X_y})$$

is surjective, and by induction, $|(\mathcal{F}_2 \otimes a_X^* P)|_{X_y}|$ induces a birational map⁶.

At this point, we have verified all the conditions in Lemma 2.1, so we may conclude that $\mathcal{F}_2 \otimes P$ induces a birational map for all $P \in \text{Pic}^0(X)$, and since $\mathcal{F}_2 = \omega_X^3 \otimes \tau(|2\Delta_X|)$, so does $\omega_X^3 \otimes P$.

5.3.2.2 Case in which $\mathcal{H}^0(\Lambda_Y)$ has torsion

If $\mathcal{H}^0(\Lambda_Y)$ has torsion, let $\hat{B}_2 \subset \hat{B}$ be a topdimensional irreducible component of the closure of the set of torsion points of $\mathcal{H}^0(\Lambda_Y)$.

If $\dim B_2 > 0$, we may consider the extended commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{a_X} & A_X \\ \downarrow f_2 & \begin{array}{c} \downarrow f_{0,1} \quad \downarrow \pi_{0,1} \\ \downarrow f_{1,2} \quad \downarrow \pi_{1,2} \end{array} & \downarrow \pi_2 \\ Y_1 & \xrightarrow{b_{Y_1}} & B_1 \\ \downarrow f_2 & \downarrow b_{Y_2} & \downarrow \\ Y_2 & \xrightarrow{\quad} & B_2 \end{array}$$

We may proceed iteratively by looking at the torsion of the sheaves $\mathcal{H}^0(\Lambda_{Y_k})$: if any of the sheaves $\mathcal{H}^0(\Lambda_{Y_k})$ is torsion-free, then we may argue as in section 5.1 with the diagram

$$\begin{array}{ccc} X & \xrightarrow{a_X} & A_X \\ \downarrow f_k & & \downarrow \pi_k \\ Y_k & \xrightarrow{b_{Y_k}} & B_k \end{array}$$

and conclude.

It thus suffices to handle the case in which $\dim B_k = 0$ for some k . In this case, we have

$$\begin{array}{ccc} X & \xrightarrow{a_X} & A_X \\ \downarrow f_{k-1} & \downarrow \pi_{k-1} & \\ Y_{k-1} & \xrightarrow{b_{Y_{k-1}}} & B_{k-1} \\ \downarrow f_{k-1,k} & \downarrow \pi_{k-1,k} & \\ Y_k & \xrightarrow{b_{Y_k}} & B_k = \{w_k\} \end{array}$$

⁶More concretely, by induction, we have that $\mathcal{F}_2^{X_y} \otimes P$ induces a birational map for any $P \in \text{Pic}^0(X_y)$ and $\mathcal{F}_2^{X_y} \otimes P = \omega_{X_y}^3 \otimes \tau(|2\Delta_{X_y}|) \otimes P$, so $(\mathcal{F}_2 \otimes P)|_{X_y} = \omega_{X_y}^3 \otimes \tau(|2\Delta_X|)|_{X_y} \otimes P|_{X_y}$ induces a birational map too.

We show that in this case, Y_{k-1} must be of general type (despite having torsion), so we may still conclude as in section 5.1. By Remark 73, if $U \subseteq \text{Pic}^0(Y_{k-1})$ is an open subset containing all the torsion points of $\mathcal{H}^0(\Lambda_{Y_{k-1}})$, then the evaluation map

$$\bigoplus_{\alpha \in U} H^0 \left(\varprojlim F_*^e b_{Y_{k-1},*} \omega_{Y_{k-1}} \otimes P_\alpha \right) \otimes P_\alpha^\vee \rightarrow \varprojlim F_*^e b_{Y_{k-1},*} (\omega_{Y_{k-1}})$$

is surjective so in particular

$$\text{rk } \mathcal{H}^0(\Lambda_{Y_{k-1}}) = \dim H^0 \left(\varprojlim F_*^e b_{Y_{k-1},*} (\omega_{Y_{k-1}}) \otimes P_\alpha \right) > 0$$

and hence Y_{k-1} must be of general type (c.f. first paragraph in section 5.3.2.1).

5.3.2.3 Case in which Y is not smooth

If Y in the Stein factorization of $\pi \circ a$ is not smooth, let Y' be a regular variety together with an alteration $b' : Y' \rightarrow Y$ and consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{a} & A \\ f \downarrow & & \downarrow \pi \\ Y' & \xrightarrow{b'} Y \xrightarrow{b} & B \\ & \searrow \beta & \end{array}$$

Arguing as in the previous subsection, we may assume that $\mathcal{H}^0(\Lambda_{Y'})$ is torsion-free, and hence that Y' is of general type. Indeed, if that was not the case, we could apply a descent argument as before and obtain a diagram

$$\begin{array}{ccccc} & X & \xrightarrow{a_X} & A_X & \\ & \downarrow & & \downarrow & \\ Y'_{k-1} & \longrightarrow & Y_{k-1} & \xrightarrow{b_{Y_{k-1}}} & B_{k-1} \\ & & \downarrow & & \downarrow \\ Y'_k & \longrightarrow & Y_k & \xrightarrow{b_{Y_k}} & B_k \end{array}$$

where $\mathcal{H}^0(\Lambda_{Y'_k})$ is either torsion-free or has torsion concentrated on a zero-dimensional set, so that either Y'_k or Y'_{k-1} are varieties of general type, and we are hence reduced to that case.

Back to the setting described in the first diagram within this subsection, we may thus assume that Y' is of general type, and we may hence consider the sheaves $F_*^e \mathcal{F}_{r,e}^{Y'} \rightarrow \mathcal{F}_r^{Y'}$ defined above. We seek to construct a sheaf on Y satisfying the conditions of Lemma 64.

Modulo complications associated with the verification of condition 1 of Lemma 64, computations analogous to those in the previous subsections would show that $b'_* \mathcal{F}_2^{Y'}$ does the job.

APPENDIX

FROBENIUS-STABLE GRAUERT RIEMENSCHNEIDER

A.1 Introduction

This appendix contains an approach to proving a Frobenius-stable Grauert Riemenschneider statement. In general, one needs very restrictive assumptions for the purpose of retrieving even the most modest versions of Kodaira or Grauert Riemenschneider vanishing theorems. For instance, in 1992, Mehta-van der Kallen proved the following relative vanishing statement in the presence of a Frobenius splitting:

Theorem 1. (c.f. [27]) *Let $f : X \rightarrow Y$ be a proper morphism of schemes. Let D (resp. E) be a closed subscheme of X (resp. Y), and let $i \geq 1$ be such that:*

1. *D contains $f^{-1}(E)$ (set-theoretically).*
2. *$R^i f_*(\mathcal{I}_D)$ vanishes outside E .*
3. *X is split compatibly with D .*

Then $R^i f_(\mathcal{I}_D) = 0$ everywhere.*

This statement can then be used in order to recover Grauert Riemenschneider and Kawamata-Viehweg vanishing theorems in the presence of a Frobenius splitting.

Theorem 2. (c.f. [7, Theorem 1.3.14]) *Let X be a nonsingular variety and $f : X \rightarrow Y$ a proper birational morphism. Assume that there exists $\sigma \in H^0(X, \omega_X^{-1})$ such that*

1. *σ^{p-1} splits X .*
2. *σ vanishes identically on the exceptional locus of f .*

Let D be an effective subdivisor of $(\sigma)_0$ containing the exceptional locus. Then $R^i f_ \mathcal{O}_X(-D) = 0$ for all $i > 0$. In particular, $R^i f_* \omega_X = 0$ for all $i > 0$.*

The proof of Theorem 1 relies on interchanging two projective limits, one involving iterated Frobenius maps and the other coming from Grothendieck's theorem on formal

functions. More concretely, assuming for a contradiction that $R^i f_*(\mathcal{I}_D) \neq 0$, the splitting assumption implies that the former projective limit is nonzero, while conditions (i) and (ii), aided by the theorem on formal functions, imply that this is not possible.

Our goal in this appendix is to present a proof of the following statement:

Theorem 3. *Let $f : X \rightarrow Y$ be a proper birational morphism of schemes, with X non-singular and Y normal, over a field of positive characteristic and assume that there is a section in $H^0(X, \omega_X^{-1})$ which vanishes identically along the exceptional locus of f . Then $\varprojlim_e R^i f_*(F_*^e \omega_X) = 0$ for all $i > 0$.*

It is worth remarking that we do not require Y to be of finite type: the proof actually reduces to the case of a complete base Y , which is stronger than what one might expect in view of the known statements in characteristic zero.

Our strategy is similar in nature to Mehta-van der Kallen's proof of Theorem 1, albeit technicalities arise when setting up our projective system. Specifically, we assemble a double projective system employing Frobenius trace maps and adjoint morphisms $F^* F_*(\bullet) \rightarrow \bullet$ and then we argue by contradiction: if $\varprojlim_e R^i f_*(F_*^e \omega_X)$ did not vanish, we show that the double projective limit would be non-zero, and then arguing via the theorem on formal functions as in [27], we conclude that this cannot possibly be the case.

Once Theorem 3 is proved, a number of obvious corollaries can be derived easily. Using the Stein factorization, one can establish an analogous vanishing statement for generically finite morphisms.

Corollary 4. *Let $f : X \rightarrow Y$ be a generically finite morphism, with X nonsingular and Y normal, over an algebraically closed field of positive characteristic and assume that there is a section in $H^0(X, \omega_X^{-1})$ which vanishes identically along the exceptional locus of the Stein factorization of f . Then $\varprojlim_e R^i f_*(F_*^e \omega_X) = 0$ for all $i > 0$.*

A similar argument also yields a Frobenius stable Nadel vanishing statement.

Corollary 5. *Let $f : X \rightarrow Y$ be a proper birational map between smooth projective varieties of characteristic $p > 0$ and assume that there is a section in $H^0(X, \omega_X^{-1})$ which vanishes identically along the exceptional locus of f . Suppose that L is a Cartier divisor on X and let Δ be a \mathbb{Q} -divisor on X such that $L - (K_X + \Delta)$ is f -big and f -semi-ample Cartier divisor on X . Then we have*

$$\varprojlim_e R^i f_*(F_*^e \omega_X([p^e(L - K_X - \Delta)])) = 0$$

for all $i > 0$.

Finally, as in characteristic zero (c.f. [25]), for surjective morphisms, we obtain (Frobenius stable) vanishing in degrees greater than the generic fiber dimension and we describe the Frobenius stable higher direct image in degree equal to the generic fibre dimension.

Corollary 6. *Let $f : X \rightarrow Y$ be a proper surjective morphism between smooth projective varieties of characteristic $p > 0$ and let $d = \dim X - \dim Y$. Then*

$$\varprojlim_e F_*^e R^i f_* \omega_X = 0$$

for every $i > d$ and

$$\varprojlim_e F_*^e R^d f_* \omega_X \simeq \varprojlim_e F_*^e \omega_Y.$$

A.2 Preliminaries

We start by recording a few results that we will be used in the sequel, most of which are taken directly from [20]. Recall that a sheaf is countably quasi-coherent if it is quasi-coherent and locally countably generated. Also recall that an inverse system of coherent sheaves $\{\Omega_e\}$ is said to satisfy the Mittag-Leffler condition if for any $e \geq 0$, the image of $\Omega_{e'} \rightarrow \Omega_e$ stabilizes for e' sufficiently large. The inverse limit functor is always left exact in the sense that if

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}_e & \longrightarrow & \mathcal{G}_e & \longrightarrow & \mathcal{H}_e \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{F}_{e-1} & \longrightarrow & \mathcal{G}_{e-1} & \longrightarrow & \mathcal{H}_{e-1} \longrightarrow 0 \end{array}$$

is an exact sequence of inverse systems, then

$$0 \rightarrow \varprojlim \mathcal{F}_e \rightarrow \varprojlim \mathcal{G}_e \rightarrow \varprojlim \mathcal{H}_e$$

is exact in the category of quasi-coherent sheaves. By a theorem of Roos (c.f. [20, Proposition I.4.1]), the right-derived functors $R^i \varprojlim = 0$ for $i \geq 2$. Hence, we have a long exact sequence

$$0 \rightarrow \varprojlim \mathcal{F}_e \rightarrow \varprojlim \mathcal{G}_e \rightarrow \varprojlim \mathcal{H}_e \rightarrow R^1 \varprojlim \mathcal{F}_e \rightarrow R^1 \varprojlim \mathcal{G}_e \rightarrow R^1 \varprojlim \mathcal{H}_e \rightarrow 0.$$

We start by recording a characterization of the Mittag-Leffler condition in terms of the first right-derived inverse limit.

Lemma 7. (c.f. [20, Proposition I.4.9]) Let $\{\Omega_e\}_e$ be an inverse system of countably quasi-coherent sheaves on a scheme X of finite type. Then the following conditions are equivalent:

1. $\{\Omega_e\}_e$ satisfies the Mittag-Leffler condition.
2. $R^1 \varprojlim \Omega_e = 0$
3. $R^1 \varprojlim \Omega_e$ is countably quasi-coherent.

The following is basic result about the cohomology of an inverse system of sheaves:

Theorem 8. (c.f. [20, Theorem I.4.5]) Let $\{\Omega_e\}$ be an inverse system of coherent sheaves on a variety X . Let T be a functor on $D(X)$ which commutes with arbitrary direct products. Suppose that $\{\Omega_e\}$ satisfies the Mittag-Leffler condition. Then for each i , there is an exact sequence

$$0 \rightarrow R^1 \varprojlim R^{i-1} T(\Omega_e) \rightarrow R^i T(\varprojlim \Omega_e) \rightarrow \varprojlim R^i T(\Omega_e) \rightarrow 0.$$

In particular, if for some i , $\{R^{i-1} T(\Omega_e)\}$ satisfies the Mittag-Leffler condition, then by Lemma 7, we have

$$R^i T(\varprojlim \Omega_e) \cong \varprojlim R^i T(\Omega_e).$$

We will be applying this theorem to the push-forward f_* under a proper morphism of schemes. We also record the following simple lemma, which roughly states that iterated inverse limits commute.

Lemma 9. Let $\{\Omega_{e,t}\}_{e,t}$ be a set of sheaves with $(e,t) \in \mathbb{Z} \times \mathbb{Z}$ ordered so that $(e,t) \leq (e',t')$ if $e \leq e'$ and $t \leq t'$. Suppose that for every $(e,t) \geq (e',t')$ there is a commutative diagram

$$\begin{array}{ccc} \Omega_{e,t} & \longrightarrow & \Omega_{e,t'} \\ \downarrow & & \downarrow \\ \Omega_{e',t} & \longrightarrow & \Omega_{e',t'} \end{array}$$

Then there are canonical isomorphisms

$$\varprojlim_e \varprojlim_t \Omega_{e,t} \simeq \varprojlim_{(e,t)} \Omega_{e,t} \simeq \varprojlim_t \varprojlim_e \Omega_{e,t}$$

Proof. This follows from the universal property of inverse limits. Note that we have maps $\varprojlim_e \Omega_{e,t} \rightarrow \Omega_{\tilde{e},t} \rightarrow \Omega_{\tilde{e},t'}$ for every $\tilde{e} \in \mathbb{Z}$ and every $t' \leq t$, so by the universal property of the system $\{\Omega_{e,t'}\}_e$, we have a map $\varprojlim_e \Omega_{e,t} \rightarrow \varprojlim_e \Omega_{e,t'}$ and analogously, we have

$\varprojlim_t \Omega_{e,t} \rightarrow \varprojlim_t \Omega_{e',t}$ for any $e' \leq e$. We may thus consider the inverse limits of the systems $\{\varprojlim_e \Omega_{e,t}\}_t$ and $\{\varprojlim_e \Omega_{e,t}\}_e$, namely $\varprojlim_t \varprojlim_e \Omega_{e,t}$ and $\varprojlim_e \varprojlim_t \Omega_{e,t}$, respectively.

In order to show that $\varprojlim_t \varprojlim_e \Omega_{e,t} \simeq \varprojlim_{(e,t)} \Omega_{e,t}$ one simply verifies that they both satisfy the same universal property. Let \mathcal{G} be any sheaf and consider a collection of maps $\varphi_{e,t} : \mathcal{G} \rightarrow \Omega_{e,t}$ commuting with the maps in the system. For fixed t , the universal property of the system $\{\Omega_{e,t}\}_e$ yields a map $\varphi_t : \mathcal{G} \rightarrow \varprojlim_e \Omega_{e,t}$, and now the universal property for the system $\{\varprojlim_e \Omega_{e,t}\}_t$ yields a map $\varphi : \mathcal{G} \rightarrow \varprojlim_t \varprojlim_e \Omega_{e,t}$, so $\varprojlim_t \varprojlim_e \Omega_{e,t}$ and $\varprojlim_{(e,t)} \Omega_{e,t}$ satisfy the same universal property and they must hence be isomorphic. One argues identically for $\varprojlim_e \varprojlim_t \Omega_{e,t}$. \square

Next we describe the behavior of inverse limits of Mittag-Leffler systems with respect to localization. To this end, we first recall the following result.

Lemma 10. (c.f. [20, Proposition I.4.7]) *Let X be a scheme of finite type over a field k , let \mathcal{F} be a countably quasi-coherent sheaf on X and let $\{\mathbb{H}_n\}$ be a descending sequence of quasi-coherent subsheaves of \mathcal{F} . Assume that the natural map $\mathcal{F} \rightarrow \varprojlim \mathcal{F}/\mathbb{H}_n$ is surjective. Then the sequence $\{\mathbb{H}_n\}$ is eventually constant.*

Using Lemma 10, we can now show that the inverse limit of a system of finitely-generated modules satisfying the Mittag-Leffler condition commutes with taking localizations.

Lemma 11. *Let R be a ring and let $\{M_n, \varphi_n\}_{n \in \mathbb{N}}$ be a inverse system of finitely generated modules which satisfies the Mittag-Leffler condition. Then for any prime ideal $\mathfrak{p} \in \text{Spec } R$, we have that the natural map*

$$(\varprojlim M_n)_{\mathfrak{p}} \longrightarrow \varprojlim M_{n,\mathfrak{p}}$$

is an isomorphism.

Proof. Let $M'_e \subset M_e$ be the image of the natural map $\varprojlim_e M_e \rightarrow M_e$ and let H_e be the kernel, so that for every e , we have a short exact sequence

$$0 \longrightarrow H_e \longrightarrow \varprojlim_e M_e \longrightarrow M'_e \longrightarrow 0$$

Since the system $\{M_e\}_e$ satisfies the Mittag-Leffler condition, we have $\varprojlim_e M_e = \varprojlim_e M'_e$ and the sequence of submodules $\{H_e\}$ is clearly decreasing. By Lemma 10, the sequence $\{H_e\}$ eventually stabilizes so that $\varprojlim M'_e = M'_\ell$ for $\ell \gg 0$. Now consider the sequences

$$0 \longrightarrow M'_e \longrightarrow M_e \longrightarrow M_e/M'_e \longrightarrow 0$$

Localizing and taking inverse limits yields

$$0 \longrightarrow \varprojlim (M'_{e,p}) \longrightarrow \varprojlim (M_{e,p}) \longrightarrow \varprojlim ((M_e/M'_e)_p)$$

But by the previous observation, $\varprojlim M'_e = M'_\ell$ for ℓ large enough, so the first system $\{M'_{e,p}\}_e$ is eventually constant and equal to $M'_{\ell,p}$ for some $\ell \gg 0$, which in turn is the same as $\left(\varprojlim_e M'_e\right)_p$ and we thus obtain an injection $\left(\varprojlim_e M_e\right)_p \hookrightarrow \varprojlim (M_{e,p})$.

Now let $Q_e = M_e/M'_e$, which satisfies the Mittag-Leffler condition (since $\{M_e\}_e$ does and $M_e \twoheadrightarrow M_e/M'_e$) and $\varprojlim Q_e = 0$ so that $\{Q_e\}_e$ is essentially zero, and hence, so is the system $\{Q_{e,p}\}$. Therefore $\varprojlim_e Q_{e,p} = 0$ and the above map is actually a bijection. \square

We record one final result for future reference.

Lemma 12. *Let R be a ring and let $\{M_n, \varphi_n\}_{n \in \mathbb{N}}$ be a projective system of finitely generated modules which satisfies the Mittag-Leffler condition. Then $\varprojlim M_n$ is finitely generated.*

Proof. Arguing as in the proof of Lemma 11, we may write $\varprojlim M_n = M'_k$ for $k \gg 0$, whence the claim follows. \square

A.3 Proof of Theorem 3

We present the theorem statement once again for convenience.

Theorem 13. *Let $f : X \rightarrow Y$ be a proper birational morphism, with X nonsingular and Y normal, both defined over a field of characteristic $p > 0$, and assume that there is a section in $H^0(X, \omega_X^{-1})$ which vanishes identically along the exceptional locus of f . Then*

$$\varprojlim_e R^i f_*(F_*^e \omega_X) = 0$$

for all $i > 0$.

Proof. The question being local on Y , we may assume that $Y = \text{Spec } R$ and it suffices to show that $\varprojlim H^i(X, F_*^e \omega_X) = 0$. Assume for a contradiction that $\varprojlim H^i(X, F_*^e \omega_X) \neq 0$ and let $y \in Y$ be the generic point of an irreducible component of its support. By Lemma 11, we have that the y lies in the support of $H^i(X, F_*^e \omega_X)$ for every $e \gg 0$, so in particular y lies in the image of the exceptional locus.

For any morphism $u : V \rightarrow W$ and any sheaves \mathcal{F}, \mathcal{G} on V, W respectively, there are natural maps $u^* u_* \mathcal{F} \rightarrow \mathcal{F}$ and $\mathcal{G} \rightarrow u_* u^* \mathcal{G}$. In particular, for the absolute Frobenius $F : X \rightarrow X$ we have $F^* F_* \omega_X \rightarrow \omega_X \rightarrow F_* F^* \omega_X$. Since X is non-singular, the Frobenius morphism is flat,

and by flat base change (c.f. [21, Proposition III.9.3]), we have $F^*F_*\omega_X \simeq F_*F^*\omega_X$ (c.f. [3, Lemma 2.2.1])¹.

Consider the following commutative diagram

$$\begin{array}{ccccccc}
 \dots & F_*^2\omega_X & \longrightarrow & F_*\omega_X & \xrightarrow{Tr_X} & \omega_X & \\
 & \uparrow & & \uparrow & & \uparrow & \\
 \dots & F^*F_*(F_*\omega_X) \simeq F_*^2(F^*\omega_X) & \longrightarrow & F^*F_*\omega_X \simeq F_*(F^*\omega_X) & & & \\
 & \uparrow & & \uparrow & & \uparrow & \\
 F_*^e F^{t,*}\omega_X \dashrightarrow \dots & F^{2,*}F_*^2(F_*\omega_X) \simeq F_*^3(F^{2,*}\omega_X) & \longrightarrow & F^{2,*}F_*^2\omega_X \simeq F_*^2(F^{2,*}\omega_X) & & &
 \end{array}$$

where the vertical maps are the natural maps $F^*F_*(\bullet) \rightarrow \bullet$, the horizontal maps are induced by the Frobenius trace map, and where the isomorphisms come from successive applications of flat base change (c.f. [5, Proposition 8.0.6]).

Commutativity is clear: in general if $j : X \rightarrow Y$ is any essentially étale morphism of schemes (e.g. the Frobenius morphism), since for any open set $U \subseteq X$, we have a map²

$$j^*j_*(F_*\omega_X)(U) = \varinjlim_{V \supset j(U)} F_*\omega_X(j^{-1}(V)) \longrightarrow \varinjlim_{V \supset j(U)} \omega_X(j^{-1}(V)) = j^*j_*\omega_X(U)$$

there is a naturally commutative diagram

$$\begin{array}{ccc}
 F_*\omega_X(U) & \xrightarrow{Tr(U)} & \omega_X(U) \\
 \uparrow j^{-1}(V) \supseteq U & & \uparrow \\
 \varinjlim_{V \supset j(U)} F_*\omega_X(j^{-1}(V)) & \xrightarrow{\varinjlim_{V \supset j(U)} Tr(j^{-1}(V))} & \varinjlim_{V \supset j(U)} \omega_X(j^{-1}(V))
 \end{array}$$

The idea is to prove that³

$$\varprojlim_e \varprojlim_t H^i(X, F_*^e F^{t,*}\omega_X)_y \neq 0 \quad (\text{A.1})$$

¹If $j : U \rightarrow X$ is a finite flat morphism of schemes (this holds more generally for essentially étale morphisms), then

$$\begin{array}{ccc}
 U & \xrightarrow{j} & X \\
 F \downarrow & & \downarrow F \\
 U & \xrightarrow{j} & X
 \end{array}$$

is a Cartesian square.

²The map is simply $\varinjlim_{V \supset j(U)} Tr(j^{-1}(V))$.

³Note that the e's increase horizontally and that both the e's and t's increase vertically. Note, however, that when taking cohomology, the inverse limit of the modules in the k-th column is isomorphic to $\varprojlim_t H^i(X, F_*^k F^{t,*}\omega_X)_y$.

and we shall then derive a contradiction by using the theorem on formal functions (as in [27]) to show that this cannot possibly be the case.

Granting (A.1) for the moment, we conclude the proof. We seek to show that for each fixed e , we have

$$\varprojlim_t H^i(X, F_*^e F^{t,*} \omega_X)_y \simeq \varprojlim_t H^i(X, F^{t,*} \omega_X)_y = 0.$$

By the theorem on formal functions (c.f. [21, Theorem III.11.1, Remark III.11.1.1]), we have an isomorphism

$$H^i(X, F^{t,*} \omega_X)_y^\wedge \simeq \varprojlim_s H^i(X_s, \nu_s^* F^{t,*} \omega_X)$$

where X_s is the s -thickened fibre over y and ν_s is the first projection in the cartesian diagram

$$\begin{array}{ccc} X_s & \xrightarrow{\nu_s} & X \\ f'_s \downarrow & & \downarrow f \\ \text{Spec } \mathcal{O}_{Y,y}/\mathfrak{m}_y^s & \longrightarrow & Y \end{array}$$

Actually, we may assume that $\mathcal{O}_{Y,y}$ is complete: indeed, choose a field of representatives K for the completion $\hat{\mathcal{O}}_{Y,y}$, so that there exist $t_1, \dots, t_n \in \mathfrak{m}_y$ such that the natural map $K[[t_1, \dots, t_n]] \rightarrow \hat{\mathcal{O}}_{Y,y}$ is surjective and the modules $H^i(X, \Omega_{e,t})$ become modules over the complete regular local ring $R := K[[t_1, \dots, t_n]]$ under the above homomorphism and we may hence argue over R . By the theorem of formal functions and using the notation described above, we then have isomorphisms

$$H^i(X, F^{t,*} \omega_X)_y \simeq \varprojlim_s H^i(X_s, \nu_s^* F^{t,*} \omega_X)$$

Hence, our limit becomes

$$\varprojlim_t H^i(X, F^{t,*} \omega_X)_y \simeq \varprojlim_t \varprojlim_s H^i(X_s, \nu_s^* F^{t,*} \omega_X) \simeq \varprojlim_s \varprojlim_t H^i(X_s, \nu_s^* F^{t,*} \omega_X) \quad (\text{A.2})$$

where the second isomorphism follows from Lemma 9. Since by assumption, there exists a global section $\sigma \in H^0(X, \omega_X^{-1})$ vanishing along the exceptional locus of f , we may write $\omega_X \simeq \mathcal{O}_X(-D) = \mathcal{I}_D$, where $D = (\sigma)_0$ is the divisor of zeros of σ . Also, from the set-theoretic inclusion $f^{-1}(E) \subset D$, it follows that $\mathcal{I}_D^t \subset \mathfrak{m}_y \cdot \mathcal{O}_X$ for some $t \gg 0$. Therefore, for every fixed s , there exists $t(s) > 0$ such that $\mathcal{I}_D^t \subset \mathfrak{m}_y^s \mathcal{O}_X$ for all $t \geq t(s)$, so in particular $\nu_s^* F^{t,*} \omega_X = \nu_s^* \mathcal{I}_D^t = 0$ and hence $\varprojlim_t H^i(X_s, \nu_s^* F^{t,*} \omega_X) = 0$, which by (A.2) implies that $\varprojlim_t H^i(X, F^{t,*} \omega_X)_y = 0$, as claimed.

This concludes the proof of the theorem provided that

$$\varprojlim_t \varprojlim_e H^i(X, F^{t,*} F_*^e \omega_X)_y \neq 0$$

which we prove in Lemma 14 below. \square

The proof of Theorem 3 will be complete once we prove the following lemma.

Lemma 14. *Let X be a nonsingular projective variety over a field of positive characteristic.*

If $\varprojlim_e H^i(X, F_^e \omega_X) \neq 0$, then*

$$\varprojlim_t \varprojlim_e H^i(X, F^{t,*} F_*^e \omega_X) \neq 0.$$

Likewise, one has $\varprojlim_t \varprojlim_e H^i(X, F^{t,} F_*^e \omega_X)_y \neq 0$.*

Proof. Consider the projective system

$$\{H^i(X, F^{t,*} F_*^e \omega_X)\}_{(e,t)}.$$

Our conclusion will follow from the following two claims:

(i) The system

$$\left\{ \varprojlim_e H^i(X, F^{t,*} F_*^e \omega_X) \right\}_t$$

satisfies the Mittag-Leffler condition.

(ii) The maps

$$\varprojlim_e H^k(X, F^{t,*} F_*^e \omega_X) \longrightarrow \varprojlim_e H^k(X, F_*^e \omega_X)$$

are non-zero for every t .

We start by showing (i) and we denote $\Omega_{e,t} = F_*^e F^{t,*} \omega_X$ in order to ease the notation. The system $\{H^i(X, \Omega_{e,t})\}_{(e,t)}$ clearly satisfies the Mittag-Leffler condition, since all the modules have finite length. By Lemmas 9 and 7, we have

$$R^1(\varprojlim_t \varprojlim_e)(H^i(X, \Omega_{e,t})) = R^1 \varprojlim_{t,e} H^i(X, \Omega_{e,t}) = 0$$

Grothendieck's spectral sequence for a composition of functors⁴ reads in this case

$$R^i \varprojlim_t (R^j \varprojlim_e H^i(X, \Omega_{e,t})) \implies R^{i+j}(\varprojlim_t \varprojlim_e)(H^i(X, \Omega_{e,t}))$$

In particular for $i+j=1$, since $\varprojlim_t \left(R^1 \varprojlim_e H^i(X, \Omega_{e,t}) \right) = 0$ (again, since for every t , the system $\{H^i(X, \Omega_{e,t})\}_e$ is Mittag-Leffler) and the differential $d_2^{1,0}$ is trivial, we have that

$$0 = R^1(\varprojlim_t \varprojlim_e)(H^i(X, \Omega_{e,t})) \simeq R^1 \varprojlim_t \left(\varprojlim_e H^i(X, \Omega_{e,t}) \right)$$

⁴It is clear that \varprojlim_e sends injectives to \varprojlim_t -acyclics, since the inverse limit of a system of injective modules is injective (by the universal property).

But since by Lemma 12, we have that $\left\{ \varprojlim_e H^i(X, \Omega_{e,t}) \right\}_t$ is countably quasi-coherent for every t , this in turn implies by Lemma 7 that $\left\{ \varprojlim_e H^i(X, \Omega_{e,t}) \right\}_t$ is Mittag-Leffler, as claimed.

We now show (ii), namely that the maps

$$\varprojlim_e H^i(X, F^{t,*} F_*^e \omega_X) \longrightarrow \varprojlim_e H^i(X, F_*^e \omega_X)$$

are non-zero for every t . We will do it for $t = 1$, the extension to arbitrary t being straight-forward. We start by noting that the vertical maps in the commutative diagram we are using are split (on account of the finiteness of F).

$$\begin{array}{ccccc} F_*^{e'} \omega_X & \xrightarrow{F_*^{e'} Tr^e} & F_*^e \omega_X & \xrightarrow{Tr^e} & \omega_X \\ \uparrow \text{dotted} & & \uparrow \text{dotted} & & \uparrow \text{dotted} \\ F^* F_* (F_*^{e'} \omega_X) & \xrightarrow{F^* F_* (F_*^{e'} Tr^e)} & F^* F_* (F_*^e \omega_X) & \xrightarrow{F^* F_* Tr^e} & F^* F_* \omega_X \end{array}$$

$$\begin{array}{ccccc} F_*^{e'} \omega_X(U) & \xrightarrow{F_*^{e'} Tr^e(U)} & F_*^e \omega_X(U) & \xrightarrow{Tr^e(U)} & \omega_X(U) \\ \uparrow \text{dotted} & & \uparrow \text{dotted} & & \uparrow \text{dotted} \\ \varinjlim_{V \supseteq U} F_*^{e'} \omega_X(V) & \xrightarrow{\varinjlim_{V \supseteq U} F_*^{e'} Tr^e(V)} & \varinjlim_{V \supseteq U} F_*^e \omega_X(V) & \xrightarrow{\varinjlim_{V \supseteq U} Tr^e(V)} & \varinjlim_{V \supseteq U} \omega_X(V) \end{array}$$

where the (upward) vertical maps are given by the universal property of the direct limit via the restrictions $\omega_X(V) \rightarrow \omega_X(U)$ for $V \supseteq U$. The splitting grounds on the fact that the flat base change isomorphism can be made explicit as follows: in general, if $j : X \rightarrow Y$ is any essentially étale morphism of schemes and $U \subseteq X$ is an open subset, we have

$$j^* j_* (F_* \omega_X)(U) = \varinjlim_{V \supset j(U)} F_* \omega_X(j^{-1}(V)) \longrightarrow F_* \omega_X(U) \longrightarrow \varinjlim_{V \supset j j^{-1} U} F_* \omega_X(V) = j_* j^* (F_* \omega_X)(U)$$

where the first map is induced by the universal property of the direct limit, via the restrictions from $j^{-1}(V)$ to $j^{-1}(j(U)) \supseteq U$, and where the second map is the natural map associated to the inclusion $U \supseteq j(j^{-1}(U))$. If j is finite (*e.g.* the Frobenius map) this boils down to

$$\varinjlim_{V \supset U} F_* \omega_X(V) \longrightarrow F_* \omega_X(U) \longrightarrow \varinjlim_{V \supseteq U} F_* \omega_X(V)$$

The commutative diagram above induces a factorization of the cohomology trace map

$$\varprojlim_e H^i(X, F_*^e \omega_X) \longrightarrow \varprojlim_e H^i(X, F^* F_* (F_*^e \omega_X)) \longrightarrow H^i(X, F^* F_* (F_*^{e'} \omega_X)) \longrightarrow H^i(X, F_*^{e'} \omega_X)$$

for every e' , and then the universal property of the inverse limit induces our sought-for non-zero map

$$\varprojlim_e H^i(X, F^* F_*(F_*^e \omega_X)) \longrightarrow \varprojlim_e H^i(X, F_*^e \omega_X)$$

□

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